CHANNEL ASSIGNMENT ON CAYLEY GRAPHS

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Abstract. We address various *channel assignment problems* on the Cayley graphs of certain groups, computing the frequency spans by applying group theoretic techniques. In particular, we show that if G is the Cayley graph of an n-generated group Γ with a certain kind of presentation, then $\lambda(G; k, 1) = 2(k + n - 1)$. For certain values of k this is the obvious optimal value for any $2n$-regular graph. A large number of groups (for instance, even Artin groups and a number of Baumslag-Solitar groups) satisfy this condition.

1. Introduction

Recall that *channel assignment problems* arise from the following setting: imagine that we are told the relative positions of a collection $\{T_1, T_2, ..., T_n\}$ of radio transmitters to which we must assign channels or frequencies $f_i = f(T_i)$ in such a fashion that if two transmitters are close enough to one another, their frequencies differ by a predetermined amount. We are then asked to find the overall narrowest (and therefore most economical) range of frequencies from which a suitable assignment of frequencies may be made. Such problems in the plane were first suggested by Hale in [6]; they have been heavily investigated in the nearly three decades since this work.

This planar problem has a natural graph-theoretical analogue, introduced by Griggs and Yeh in [5]. In the graph setting, the vertices V of a simple graph G represent the transmitters, and the distance between two vertices is measured by the usual path metric ρ induced by the graph's edge set E. For instance, we may be asked to find the smallest value λ such that there exists a vertex labeling $\ell : V \to \{0, 1, ..., \lambda\}$ satisfying $|\ell(u) - \ell(v)| \geq k_i$ if $\rho(u, v) = i$, for $i = 1, 2$. Such a labeling is called a $L(k_1, k_2)$-*labeling*. An enormous amount of work has been done in computing this *span* $\lambda(G; k_1, k_2)$ in various settings. See [1] for a comprehensive survey of relevant work.

Note. Sometimes $\lambda(G; k_1, k_2)$ is defined to be the minimum over all valid labelings $\ell : V \to \mathbb{N}$ of the difference $\max_{v\in V} \ell(v) - \min_{v\in V} \ell(v)$. However, without loss of generality we may assume that $\min_{v\in V} \ell(v) = 0$, and we make this assumption throughout the sequel.

The present work is motivated in part by the recent work [4] of Griggs and Jin, in which the spans $\lambda(G; k, 1)$ are computed for the three regular lattices in the Euclidean plane, for *real* values k. Note that here we consider only integral values for k.

Our attention here is on infinite graphs arising as the undirected graphs underlying Cayley graphs of certain groups. We recall that the *Cayley graph* of the group Γ relative to the finite group generating set S is the labeled directed graph (V, E) for which $V = \Gamma$ and $E = \{(u, us) \mid u \in V, s \in S\}$, where the edge (u, us) is labeled s. That is, there is an edge labeled s between two

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vertices of G if one is obtained from the other through right multiplication by s. Note that if $|S| = n$, then the undirected graph underlying the Cayley graph G is $2n$-regular.

Channel assignment on certain Cayley graphs has been considered before, although not extensively. For instance, in [11], [12], and [2] Zhou, *et al.*, consider Cayley graphs of abelian groups. Although some of the groups we examine here are abelian, we obtain bounds on λ for many non-abelian groups as well. We note that by construction all of our labelings are "no-hole," as all values in $\{0, ..., \lambda\}$ are used.

We also note that some of the techniques we develop below are similar to those of van den Heuvel, Leese, and Shepherd [7], who consider some of the same graphs as we do here. However, our results generalize some of the familiar infinite graphs (the infinite triangular and square lattices, for instance) in a different manner than that considered in [7].

Although we will make use of the edge labels and orientations in defining our labeling, these data can be ignored once the labeling is complete, allowing us to consider the underlying undirected graph.

Let S be a set, and let S^{-1} be the set of formal inverses of elements of S, so $S^{-1} = \{s^{-1} \mid s \in S\}$. Let R be a set of *relations* over the alphabet $S^{\pm 1} = S \cup S^{-1}$. That is, each element $r \in R$ is an equation of the form $w_1 = w_2$, where w_1 and w_2 are words in the alphabet $S^{\pm 1}$. We say that the group Γ has *presentation* $\langle S \mid R \rangle$ if Γ is isomorphic to the quotient group $F(S)/N(R)$, where $F(S)$ is the free group generated by the set S, and $N(R)$ is the smallest normal subgroup of $F(S)$ generated by the relations in R. Roughly speaking, an element $\gamma \in \Gamma$ is trivial if and only if it can be written as a product of conjugates of terms $(w_1 w_2^{-1})^{\epsilon}$, $\epsilon = \pm 1$, for $w_1 = w_2$ an element of R.

Fix a generating set S and let w be a word in $S^{\pm 1}$. The *exponent sum* of s in w, denoted $\exp_s(w)$, is the sum of the exponents on occurrences of s in w (including both positive and negative exponents). For a fixed natural number $N \in \mathbb{N}$ we say that a presentation $\langle S \mid R \rangle$ is (s, N)-*balanced* if for every relation $(w_1 = w_2) \in R$, $\exp_s(w_1) \equiv \exp_s(w_2) \mod N$. We say that the presentation is N-*balanced* if it is (s, N)-balanced for every $s \in S$.

Our main result is the following

Theorem 1.1. *Let k be a fixed positive integer, and let Γ be the group given by the $\langle S \mid R \rangle$, where $n = |S|$. Suppose that $S \cap S^{-1} = \emptyset$ and that the presentation $\langle S \mid R \rangle$ is $(2(n+k)-1)$-balanced. Then for the underlying undirected graph G of the Cayley graph $G(\Gamma, S)$ we have $\lambda(G; k, 1) \leq 2(k+n-1)$, and equality holds if $k \leq 2n$.*

As we will see in the following section, if $k < d + 1$ then $\lambda(G; k, 1) \geq 2k + d - 2$ for *all* d-regular simple graphs, finite or infinite. Moreover, Section 2 will close with a slightly more general version of this theorem whose proof is entirely analogous.

As a sample application of Theorem 1.1, the integer lattice graph $G = \mathbb{Z} \times \mathbb{Z}$ is the 4-regular Cayley graph of the Artin group $A = \langle s, t \mid st = ts \rangle$ (see Section 3). We thus recapture the value $\lambda(G; 2, 1) = 6$, derived in [4] and elsewhere.

We consider other specific examples in Section 3.

2. Proof of the main theorem

We begin this section by noting that the proposed span $\lambda(G; k, 1) = 2(k + n - 1)$ is the best possible for an $2n$-regular simple graph, if k is "small." The following proposition was proven by Georges and Mauro [3]:

Proposition 2.1. *Let G be a d-regular graph, and let $k < d + 1$. Then $\lambda(G; k, 1) \geq 2k + d - 2$.*

Notes. In the case $k \leq 2$, any Cayley graph will satisfy the condition $k < d+1$, since $k \leq 2 < 2n+1$ even if G is 1-generated. To see that Proposition 2.1 may fail when $k \geq d+1$, let $d = 4$ and $k = 5$, and consider the 4-regular infinite tree G (G is the Cayley graph of the free group F_2 on 2 generators). We label G's vertices using the values $A = \{0, 1, 2, 3\}$ and $B = \{8, 9, 10, 11\}$. Label $V(G)$ so that each A-vertex is adjacent to precisely one B-vertex of each type, and analogously for each B-vertex. This yields a labeling with span $11 = 2k + d - 3 < 2k + d - 2 = 12$.

Until further notice, suppose that Γ is given by the $(2(k+n)-1)$-balanced presentation $\langle S \mid R\rangle$ with $n = |S|$ and $S = \{s_1, ..., s_n\}$, and that $G = G(\Gamma, S)$ is the corresponding $2n$-regular Cayley graph. Let $m = 2(k+n) - 1$.

We define our labeling first on the set S, by letting $\ell(s_i) = k + i - 1$ for $i = 1, ..., n$. Next, extend ℓ to S^{-1} by letting $\ell(s_i^{-1}) = m - \ell(s_i) = m - k - i + 1$ for $i = 1, ..., n$. This ℓ then extends to a group homomorphism from the free group $F(S)$ to $\mathbb{Z}_m$. Explicitly, for a word $w = s_{i_1}^{\epsilon_1} \cdots s_{i_r}^{\epsilon_r}$ ($\epsilon_j = \pm 1$ for all j) in the free group,

$$\ell(w) = \ell(s_{i_1}^{\epsilon_1} \cdots s_{i_r}^{\epsilon_r}) = \epsilon_1 \ell(s_{i_1}) + \cdots + \epsilon_r \ell(s_{i_r}) = \sum_{j=1}^{n} \exp_{s_j}(w) \ell(s_j),$$

where addition is performed in $\mathbb{Z}_m$.

Lemma 2.2. *The map $\ell : F(S) \to \mathbb{Z}_m$ induces a group homomorphism $\ell : \Gamma \to \mathbb{Z}_m$.*

Proof. This is an immediate consequence of the fact that $\langle S \mid R\rangle$ is m-balanced. By Dyck's Theorem (see [10]), we need only show that every relation r in R is preserved by the action of ℓ. Indeed, let $r = (w_1 = w_2) \in R$. Then

$$\ell(w_1) = \sum_{j=1}^{n} \exp_{s_j}(w_1) \ell(s_j) \equiv \sum_{j=1}^{n} \exp_{s_j}(w_2) \ell(s_j) = \ell(w_2),$$

where equivalence is modulo m. Thus the relation r is preserved. □

Since $V = V(G) = \Gamma$, ℓ serves as a labeling for V as well. Note that if $u, v \in V$ are adjacent, then without loss of generality $v = us$ for some $s \in S$, so that

$$\ell(v) - \ell(u) = \ell(us) - \ell(u) = \ell(s) \geq k,$$

so ℓ satisfies the first condition we must check in order that it be a valid $L(k, 1)$-labeling. For the second, note that if $\rho(u, v) = 2$, then $v = ust$ for some $s, t \in S^{\pm 1}$.

Lemma 2.3. *Let $s, t \in S^{\pm 1}$, $s \neq t^{-1}$. Then $\ell(s) + \ell(t) \not\equiv 0 \bmod m$.*

Proof. Clearly if $s, t \in S$, $\ell(s) + \ell(t) \leq 2(k+n) - 2 < m$, and if $s, t \in S^{-1}$, $m + 1 = 2(k+n) \leq \ell(s) + \ell(t) \leq 2(k + 2n - 1) = m + 2n - 1$. In either case, $\ell(s) + \ell(t) \not\equiv 0 \bmod m$.

Without loss of generality, assume $s = s_i \in S$ and $t = (s_j)^{-1} \in S^{-1}$. Then $\ell(s) + \ell(t) = i + k - 1 + m - j - k + 1 = m - j + i \neq m$, so $\ell(s) + \ell \not\equiv 0 \bmod m$ in this case too. □

Thus if $v = ust$, $\ell(v) = \ell(u) + \ell(s) + \ell(t) \neq \ell(u)$, so the labels on u and v are distinct, as desired. Thus ℓ witnesses $\lambda(G; k, 1) = m - 1 = 2(k + n - 1)$, and our main theorem is proven.

We may relax the condition $S \cap S^{-1} = \emptyset$ slightly. Let us allow $S = \{s_1, ..., s_n\}$ to contain a single involution, an element of order 2. In this case, the corresponding Cayley graph will be regular of degree $d = 2n - 1$, since at each vertex we can identify the "incoming" and "outgoing" edges labeled by the single involution.

Theorem 2.4. *Let $k \in \mathbb{N}$ be fixed. Let Γ be the group given by the presentation*

$$\langle S \cup \{t\} \mid R \cup \{t^2 = 1\}\rangle,$$

where $|S| = n-1$ and R is a $(2(k+n-1))$-balanced set of relations in $(S \cup \{t\})^{\pm 1}$. Suppose that $S \cap S^{-1} = \emptyset$. Let $G = G(\Gamma, (S \cup \{t\}))$ be the undirected $(2n-1)$-regular graph underlying the Cayley graph of Γ relative to $S \cup \{t\}$. Then $\lambda(G; k, 1) \leq 2(k+n) - 3$, and equality obtains if $k < 2n$.

Note that the purported span $2(k+n) - 3 = 2k + d - 2$, just as in Theorem 1.1.

Proof. Let $m = 2(n+k-1)$. We may let $s_n = t$, so that $s_n^2 = 1$, or $s_n = s_n^{-1}$. As before we define $\ell : \{s_1, ..., s_n\} \to \{0, ..., 2(k+n)-3\}$ by $\ell(s_i) = k + i - 1$ and $\ell(s_i^{-1}) = m - k - i + 1$. Note that $\ell(s_n) = k + n - 1$, so $\ell(s_n) = \ell(s_n^{-1})$.

Just as before this ℓ extends first to a homomorphism from $F(s_1, ..., s_n)$ to the group $\mathbb{Z}_m$, then to a homomorphism from Γ to $\mathbb{Z}_m$. Once more it is easily verified that ℓ satisfies the required separation conditions, witnessing $\lambda(G; k, 1) \leq 2(k+n) - 3$, with equality obtaining when indicated by Proposition 2.1. □

We note one more variation of the main theorem. Recall that $\ell : V \to \mathbb{N}$ is an $L(k_1, k_2)$-labeling if $\rho(u, v) = i$ implies that $|\ell(u) - \ell(v)| \geq k_i$, for $i = 1, 2$. The following is proven in exactly the same way as is the main theorem:

Theorem 2.5. *Let $k_1, k_2 \in \mathbb{N}$ be fixed, and let Γ be the group given by the $(2(k_1 + (n-1)k_2) + 1)$-balanced presentation $\langle S \mid R\rangle$, where $n = |S|$. Suppose also that $S \cap S^{-1} = \emptyset$ (that is, no generator is inverse to another). Let $G = G(\Gamma, S)$ be the undirected $2n$-regular graph underlying the Cayley graph of Γ relative to S. Then $\lambda(G; k_1, k_2) \leq 2(k_1 + (n-1)k_2)$.*

Clearly this reduces to the main theorem when we set $k_1 = k$ and $k_2 = 1$.

3. Specific examples

The examples in this section are meant to be illustrative and not exhaustive. Indeed, the method we have developed here applies to an incredibly diverse collection of Cayley graphs, and in the concluding section we will consider a generalization of the method which addresses almost all Cayley graphs in certain interesting classes. For the time being, let us indicate a few of the more important examples of graphs for which our method yields results.

Given two generators s and t of a group G and $m \in \mathbb{N}$, let $w_m(s, t)$ denote the alternating product $stst \cdots$ of length m. An *Artin group* is any group given by a presentation of the form $\langle S \mid R\rangle$ where $S = \{s_1, ..., s_n\}$ and

$$R = \{w_{m_{ij}}(s_i, s_j) = w_{m_{ij}}(s_j, s_i) \mid m_{ij} \in \mathbb{N} \cup \{\infty\},\ m_{ij} = m_{ji},\ \text{and } m_{ij} = 1 \Leftrightarrow i = j\}.$$

We take $m_{ij} = \infty$ to mean there is no relation involving s_i and s_j. S is called the *fundamental generating set* for A. We say that such A is *even* if every m_{ij} is either infinite or even whenever $i \neq j$.

Theorem 1.1 immediately implies the following

Corollary 3.1. *Let $G = G(A, S)$ be the Cayley graph of an even Artin group A, with respect to the fundamental generating set. Then $\lambda(G; k, 1) \leq 2(k + |S| - 1)$, and equality obtains if $k < 2|S| + 1$.*

For example, as mentioned in the introduction, the integer lattice graph $G = \mathbb{Z}\times\mathbb{Z}$ is the Cayley graph of $A = \langle s, t \mid st = ts\rangle$, and so $\lambda(G; k, 1) \leq 2(k+2-1) = 2k+2$. In particular, $\lambda(G; 2, 1) = 6$, $\lambda(G; 3, 1) = 8$, and $\lambda(G; 4, 1) = 10$, recovering part of the formula given in Theorem 3.4 of [4]. (That theorem gives a precise value of $\lambda(G; k, 1) = k + 6$ for $k \geq 5$.)

We may generalize $\mathbb{Z} \times \mathbb{Z}$ in a different manner. This graph corresponds to the fundamental group of the genus-1 torus. More generally, the 0-balanced (and therefore N-balanced, for any N) one-relator presentation

$$\langle s_1, s_2, ..., s_n, t_1, t_2, ..., t_n \mid s_1t_1s_1^{-1}t_1^{-1} \cdots s_nt_ns_n^{-1}t_n^{-1} = 1\rangle$$

gives the fundamental group of a genus-n torus, given $n \in \mathbb{N}$. Theorem 1.1 shows that the Cayley graph G of this group relative to the given generating set has span $\lambda(G; k, 1) \leq 2(k+2n-1)$, with equality obtaining whenever $k < 4n + 1$.

Another interesting family of groups are the *Baumslag-Solitar groups*; $\mathrm{BS}(1, r)$ is given by the presentation $\langle a, b \mid ab = ba^r\rangle$. These groups arise frequently in applications of group theory to geometry and topology, and are canonical examples of highly "non-hyperbolic" groups. The following is an immediate consequence of the main theorem:

Corollary 3.2. *Let k be fixed and let $r \equiv 1 \bmod 2k+3$. If G is the Cayley graph of the Baumslag-Solitar group $\mathrm{BS}(1, r)$, then $\lambda(G; k, 1) \leq 2k + 2$, with equality for $k < 5$.*

As a sample application of Theorem 2.4, consider the infinite *ladder* L_∞, two infinite paths with corresponding vertices connected by a single edge. L_∞ is the Cayley graph corresponding to the group presentation $\langle s, t \mid st = ts, t^2 = 1\rangle$. Theorem 2.4 applies and shows that $\lambda(L_\infty; k, 1) \leq 2(k + n) - 3 = 2k + 1$, with equality whenever $k < 4$. Thus $\lambda(L_\infty; 2, 1) = 5$ and $\lambda(L_\infty; 3, 1) = 7$.

Note that the span λ is "weakly inherited" by subgraphs: if $H \leq G$, $\lambda(H; k, 1) \leq \lambda(G; k, 1)$. Using this property of λ we obtain the following

Corollary 3.3. *Let L be any ladder, finite or infinite. Then $\lambda(L; k, 1) \leq 2k + 1$, with equality obtaining whenever $k < 4$.*

By identifying the ends of a ladder we obtain a *prism* Pr_n, consisting of two cycles of length n, corresponding points on which are connected by a single edge. Note that Pr_n is the Cayley graph corresponding to the presentation $\langle s, t \mid s^n = 1, t^2 = 1\rangle$, with involution t. We may now apply Theorem 2.4 to obtain

Corollary 3.4. *Let k be fixed. If $n \equiv 0 \bmod 2k + 2$, then $\lambda(Pr_n; k, 1) \leq 2k + 1$, with equality obtaining when $k < 4$.*

For instance, we recover the well-known fact $\lambda(Pr_n; 2, 1) = 5$ when $n \equiv 0 \bmod 6$.

4. Generalizations and directions for further study

As indicated at the outset of the preceding section, this article has considered only a fraction of the group presentations which may induce Cayley graph labelings with conditions at various distances.

We may gain a good deal of flexibility by relaxing the rigid structure of the homomorphism by which we define our labeling.

For instance, the *unbalanced* presentation $\langle s, t \mid ts^4 = st^3\rangle$ permits an $L(2, 1)$-labeling ℓ with span $\lambda = 6$, produced by letting $\ell(s) = 2$ and $\ell(t) = 3$ and extending as in the proof of the main theorem. This optimal labeling is obtainable since these choices for $\ell(s)$ and $\ell(t)$ satisfy

$$4\ell(s) + \ell(t) \equiv \ell(s) + 3\ell(t) \bmod 7.$$

We can analyze such 2-generator, 1-relator groups more generally: without loss of generality we may write any single relator $w = w'$ as $w(w')^{-1} = 1$, so we assume our group is given by the presentation $\langle s, t \mid w = 1\rangle$. Let $e_s = \exp_s(w)$ and $e_t = \exp_t(w)$. Should it be the case that one of

$$2e_s + 3e_t, 3e_s + 2e_t, 2e_s + 4e_t, \text{ or } 4e_s + 2e_t$$

is congruent to 0 mod 7, then we may define a homomorphism ℓ which yields an $L(2,1)$-labeling with span $\lambda = 6$. One of these equivalences holds in 25 of the 49 possible choices for $(\exp_s(w), \exp_t(w))$ after reduction modulo 7. Assuming each of these 49 choices is equally likely, we have

Proposition 4.1. *Let G be the Cayley graph corresponding to a 2-generator, 1-relator presentation selected uniformly at random from among all such presentations. Then the probability that our method yields an $L(2,1)$-labeling ℓ of G with span $\lambda = 6$ is $\frac{25}{49} \approx 0.5102$.*

Indeed, the preceding result is a special case of the following much more general observation, whose proof is nearly identical to the argument in Section 2:

Proposition 4.2. *Let G be the $2n$-regular undirected graph underlying the Cayley graph corresponding to the involution-free presentation $\langle s_1, s_2, ..., s_n \mid w\rangle$ (for w a word in the letters $S^{\pm 1}$), and define e_i to be the exponent sum of s_i in w, modulo $2n+3$. Let $\vec{e} = (e_1, e_2, ..., e_n) \in \mathbb{Z}^n_{2n+3}$ be the presentation's* exponent vector*, and suppose there exists an* assignment vector $\vec{a} = (a_1, a_2, ..., a_n)$ *such that*

(1) $a_i \in \{2, 3, ..., 2n, 2n+1\}$ *for* $i = 1, ..., n$,
(2) $a_i \neq \pm a_j \bmod 2n+3$ *if* $i \neq j$, *and*
(3) $\vec{e} \cdot \vec{a} \equiv 0 \bmod 2n+3$.

Then $\lambda(G; 2, 1) = 2n + 2$.

Let us call any presentation satisfying the hypothesis of Proposition 4.2 *semibalanced*. Numerical evidence suggests that semibalanced presentations are quite common. For a given n, we let $\epsilon(n) = (2n+3)^n$ denote the number of potential exponent vectors $\vec{e}$ and $\epsilon_{sb}(n)$ denote the number of such vectors corresponding to semibalanced presentations as in Proposition 4.2.

Direct computation yields the following information for small values of n:

n	$\epsilon(n)$	$\epsilon_{sb}(n)$	$\frac{\epsilon_{sb}(n)}{\epsilon(n)}$
1	5	1	0.20000
2	49	25	0.51020
3	729	603	0.82716
4	14641	14481	0.98907
5	371293	370993	0.99919

Thus the proportion of exponent vectors corresponding to semibalanced presentations appears to converge very quickly to 1. (The rate of convergence is even quicker when only those n for which $2n + 3$ is prime are considered.) These data suggest the following

Conjecture 4.3. *The probability that the undirected graph G underlying the Cayley graph of an n-generator, one-relator presentation selected uniformly at random from among all such presentations satisfies $\lambda(G; 2, 1) = 2n + 2$ approaches 1 as $n \to \infty$.*

Proving this conjecture will require, for a given n, estimating the number of exponent vectors orthogonal (mod $2n+3$) to some assignment vector. This may be easier when $2n+3$ is prime, in which case the discrete Fourier analytic methods of Iosevich and Senger in [8] can apply. These authors have already indicated that the problem is likely a tractable one [9] and have begun collaboration with the present author in order to complete a proof of the conjecture.

To conclude we mention another generalization, this one of the graph theoretic conditions imposed on the labelings. Griggs and Yeh [5] consider conditions at p distinct distances, posing the problem of finding the span $\lambda(G; k_1, ..., k_p)$ of an optimal $L(k_1, ..., k_p)$-labeling. It is clear that though the computations will become messier, the approach undertaken in this article can be generalized to produce labelings ℓ witnessing bounds for these more general spans λ. Although there is no theoretical obstacle to this generalization, describing the desired group homomorphisms will likely become much more difficult except in very specific cases.

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