

Handout 15: Homomorphisms in general (universal) algebras

Let's start out with the following definition, at which we hinted at the end of the last class:

Definition. Let A and B be \mathcal{F} -algebras, and let $\phi : A \rightarrow B$ be a function from A to B . We say that ϕ is an \mathcal{F} -algebra homomorphism (or simply *homomorphism*, when \mathcal{F} is understood) if for every $f \in \mathcal{F}_n$ and every $a_1, a_2, \dots, a_n \in A$, we have

$$\phi(f^A(a_1, a_2, \dots, a_n)) = \phi(f^B(\phi(a_1), \phi(a_2), \dots, \phi(a_n))).$$

Isomorphisms, endomorphisms, and automorphisms are defined as usual.

Let's see what we can prove with this new definition!

Definitions/Theorem. If A is an \mathcal{F} -algebra and $S \subseteq A$ is a subset of A that is *closed* under all of the operations f^A (for $f \in \mathcal{F}$), then S can be made into a *subalgebra* of A by defining f^S to be the *restriction* of f^A to S .

Let $\phi : A \rightarrow B$ be an \mathcal{F} -algebra homomorphism. Then if C is a subalgebra of A , $\phi(C)$ is a subalgebra of B . (In particular, the *image* $\text{im}(\phi) = \phi(A)$ of ϕ is a subalgebra of B .)

Proof. We really just have to check that $D = \phi(C)$ is closed under all of the operations f^B . Go ahead:

Theorem. If $\phi : A \rightarrow B$ and $\psi : B \rightarrow C$ are \mathcal{F} -algebra homomorphisms, then so is $\psi \circ \phi : A \rightarrow C$.

Proof.

Theorem. Let A be an \mathcal{F} -algebra, let $\theta \in \text{Con}(A)$, and define the *natural map* $\nu : A \rightarrow A/\theta$ by $\nu(a) = [a]_\theta$. Then ν is a surjective homomorphism.

Proof. Follow your nose!

We'll need one more crucial

Definition. Given an \mathcal{F} -algebra homomorphism $\phi : A \rightarrow B$, the *kernel* of ϕ , denoted $\ker(\phi)$, is defined by

$$\ker(\phi) = \{(a, a') \mid \phi(a) = \phi(a')\}.$$

Theorem. If $\phi : A \rightarrow B$ is a homomorphism, then $\ker(\phi) \in \text{Con}(A)$.

Proof. Just check the definition! It's got to be an equivalence relation with "something extra"...

Here we go, folks! Are you ready?

First Homomorphism Theorem. Let $\phi : A \rightarrow B$ be an \mathcal{F} -algebra homomorphism. Then $A/\ker(\phi) \cong \text{im}(\phi)$.

Here's a picture for a nice commutative diagram:

And here's some room for a

Proof. You really just have to define the right function and show that it works!

There are a number of other cool results concerning homomorphisms and quotients which are likely accessible to you now, should you feel like perusing Burris and Sankappanavar's on-line text!

Homework. Here are a few exercises for you to think about; there's a single committee problem, marked with "(CP)."

- (1) **(CP)** Fill in the blank and prove your assertion: an \mathcal{F} -algebra homomorphism $\phi : A \rightarrow B$ is injective if and only if $\ker(\phi) = \underline{\hspace{2cm}}$. (*Hint:* think about what elements of A should be sent to the same element in B ...)
- (2) Let $\text{Aut}(A)$ denote the collection of \mathcal{F} -algebra automorphisms of A . Prove that $\text{Aut}(A)$ is a group under the operation of composition.
- (3) Suppose that A is an \mathcal{F} -algebra and that $\theta_1, \theta_2 \in \text{Con}(A)$ satisfy $\theta_1 \subseteq \theta_2$. (Note that this means $a \equiv_{\theta_1} a' \Rightarrow a \equiv_{\theta_2} a'$.) Define

$$\theta_2/\theta_1 = \{([a]_{\theta_1}, [a']_{\theta_1}) \in A/\theta_1 \times A/\theta_1 \mid (a, a') \in \theta_2\}.$$

Prove that θ_2/θ_1 is a congruence on A/θ_1 .

- (4) **Extra credit.** Prove that, with A and θ_i ($i = 1, 2$) as above,

$$(A/\theta_1)/(\theta_2/\theta_1) \cong A/\theta_2.$$

Here is a schedule for this homework:

- **Monday, March 23rd.** Committee problem due to committee.
- **Wednesday, March 25th.** Committee presents its report to class.
- **Friday, March 27th.** All homework due at 5:00 p.m.