MATH 462, Section 1
Abstract Algebra II
Spring 2009

Handout 14: Congruences in general (universal) algebras

Let's start by reviewing the essential concepts regarding universal algebras, which we discussed in class last time:

Definitions. For $n \in \mathbb{N} \cup \{0\}$, an *$n$-ary function* on a set A is a function f taking n-tuples as input and providing a single element of A as output. We're most familiar with *unary* (1-ary), *binary* (2-ary), and *nullary* (0-ary) functions, the last of which are simply constants.

Definitions. A set $\mathcal{F}$ of n-ary function symbols is called a *type*. An *algebra* of type $\mathcal{F}$ (or an $\mathcal{F}$-algebra) is a set possessing its own "versions" of the function symbols lying in $\mathcal{F}$. We denote A's version of $f \in \mathcal{F}$ by f^A. That is, if f is an n-ary function symbol in the type $\mathcal{F}$, f^A is an n-ary function on A. Typically the algebras of a certain type will be defined by means of some set of axioms indicating the way those algebras' n-ary functions interact with one another.

Examples. The two examples of typed algebras with which we're most familiar are groups and rings. In the space below you should be able to describe the types of each of these sorts of algebras in terms of their function symbols and governing axioms:

- *Groups:*

- *Rings:*

Our goal now is to generalize two constructions we've now seen on both groups and rings, and to use these constructions to derive a more general version of the First Homomorphism Theorem.

We need to know how to generalize both *homomorphisms* and *quotients*. In the types of algebras we've considered so far, we've done homomorphisms first and then quotients; however, since Burris and Sankappanavar consider them in the opposite order, we'll follow their lead and work with quotients first.

The universal algebra equivalent of "quotient" is "congruence."

Definitions. Let A be an algebra of type $\mathcal{F}$, and let $\mathrm{Eq}(A)$ denote the set of equivalence relations on A. (Thus $\mathrm{Eq}(A) \subseteq A \times A$, as usual.) Let $\theta \in \mathrm{Eq}(A)$. We say that θ is a *congruence* on A if for any n-ary function symbol $f \in \mathcal{F}_n$ and for any $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ in A,

$$(a_i, b_i) \in \theta \text{ for all } i \Rightarrow (f^A(a_1, ..., a_n), f^A(b_1, ..., b_n)) \in \theta.$$

The set of all congruences on A is denoted by $\mathrm{Con}(A)$.

Note that a congruence is simply an equivalence relation that "works well with" all n-ary operations on A.

Examples.

(1) Let G be a group (with type $\mathcal{F} = \langle \cdot, ^{-1}, 1 \rangle$) and let N be a normal subgroup of G. Define θ_N by $(g, g') \in \theta_N$ if and only if $g(g')^{-1} \in N$. Prove that $\theta \in \mathrm{Con}(G)$.

(2) Let R be a ring (with type $\mathcal{F} = \langle \cdot, +, -, 0, 1 \rangle$), and let I be an ideal in R. Define θ_I by $(r, r') \in \theta_I$ if and only if $r - r' \in I$. Prove that $\theta_I \in \mathrm{Con}(R)$.

Ready for something cool? Here's our first "universal" generalization of a substantial result:

Definition/Theorem. Let A be an algebra of type $\mathcal{F}$ and let $\theta \in \mathrm{Con}(A)$. Then the set A/θ of all equivalence classes $[a]_\theta$ under θ is an algebra of type $\mathcal{F}$, with $f^{A/\theta}$ defined by

$$f^{A/\theta}([a_1]_\theta, ..., [a_n]_\theta) = [f^A(a_1, ..., a_n)]_\theta.$$

Proof. Really all we have to do is check to see that the functions $f^{A/\theta}$ as defined are indeed well-defined. Here's some space for you to do that, relying only on the simple fact that θ is a congruence:

Notice that this is one of those instances in which we *defined* a concept in order to *force* a nice theorem to be true!

Homework. The following exercises are due on *Friday, March 20th.*

(1) Let A be an algebra of a given type $\mathcal{F}$, and let θ denote the equivalence relation defined by $(a, b) \in \theta \Leftrightarrow a = b$. Prove that θ is indeed a congruence, and describe A/θ as well as you can.

(2) Let A be an algebra of a given type $\mathcal{F}$, and let θ denote the equivalence relation $\theta = A \times A$. Prove that θ is an equivalence relation. Is it always a congruence? (Technically we don't have enough background to answer this last question; but try explaining why it may or may not be the case.) Describe A/θ as well as you can.