MATH 462, Section 1
Abstract Algebra II
Spring 2009

Ring homomorphisms and related concepts, Part I

Definitions. Not too surprisingly, a *ring homomorphism* is a function $\phi : R_1 \to R_2$ between rings $(R_1, \cdot_1, +_1)$ and $(R_2, \cdot_2, +_2)$ that preserves the structure of the ring:

1. $\phi(1_{R_1}) = 1_{R_2}$,
2. $\phi(r +_1 s) =$ ______________ for all $r, s \in R_1$, and
3. $\phi(r \cdot_1 s) =$ ______________ for all $r, s \in R_1$.

We call ϕ an ________________ if ϕ is additionally bijective. A homomorphism from a ring R to itself is called an ________________ ; if such a homomorphism is bijective, it is called an ________________ .
Let $\phi : R_1 \to R_2$ be a homomorphism. The __________ of ϕ, denoted _________ , is the set

$$\{r \in R_1 \mid \phi(r) = \_\_\_\_\_ \} \subseteq R_1.$$

The __________ of ϕ, denoted _________ , is the set

$$\{r \in R_2 \mid \phi(r_1) = r_2 \text{ for some } r_1 \in R_1\} \subseteq R_2.$$

Wow!

Examples. In each case below, do enough work to convince yourself that the desired function is a homomorphism (or more!), and determine its kernel and image.

1. Let R be a ring, and $R[x]$ its polynomial ring. What's a natural isomorphism ν between R and some subring of $R[x]$?

2. Let R be an integral domain, and let $F = \text{Frac}(R)$ be its field of fractions, $F = \{[r, s] \mid r, s \in R\}$. What's a natural isomorphism between R and some subring of F?

3. Consider the field $\mathbb{C}$ of complex numbers, and define $\phi : \mathbb{C} \to \mathbb{C}$ by $\phi(z) = \overline{z}$, the complex conjugate of $z \in \mathbb{C}$. Prove that ϕ is an automorphism, using the fact that $a + bi = \alpha + \beta i$ if and only if $a = \alpha$ and $b = \beta$:

4. Let R be a ring with identity 1_R and define $\phi : \mathbb{Z} \to R$ by $\phi(n) = n \cdot 1_R$. Prove that ϕ is a homomorphism. When is ϕ injective?

5. If R is a ring and $r \in R$, then the *evaluation at* r is the operation $e_r : R[x] \to R$ defined by

$$e_r(p(x)) = p(r)$$

for all polynomials $p(x) \in R[x]$. Convince yourself that e_r is a homomorphism:

This last example is a special case of a more general theorem we will not prove (a proof can be found on pages 244–246 of your text):

Theorem (3.33). Let R and S be rings and $\phi : R \to S$ be a homomorphism. For any $s_1, ..., s_n \in S$ there is a unique homomorphism $\tilde{\phi} : R[x_1, ..., x_n] \to S$ such that $\tilde{\phi}(x_i) = s_i$ for all $i = 1, ..., n$ and $\tilde{\phi}(r) = \phi(r)$ for all $r \in R$.

Note that this is essentially saying there's only one way to construct a commutative diagram of ring homomorphisms involving R, $R[x_1, ..., x_n]$, and S, and the homomorphisms ϕ and ν from the first example above. You can draw this diagram in the space below:

Our next handout will deal more closely with kernels of homomorphisms and, more generally, *ideals*.