Polynomial rings

Given the important place that polynomial rings hold in abstract algebra, it's worth our while to review these structures before proceeding.

Definitions. Let $(R, \cdot, +)$ be a ring with identity. A *formal power series over* R (in a single variable) is a function $\sigma : \mathbb{N}_0 \to R$. If there is some $N \in \mathbb{N}_0$ such that $\sigma(n) = 0_R$ for all $n \geq N$, σ is called a *polynomial* over R. We can write a power series as a vector rather compactly:

$$\sigma = (\sigma(0), \sigma(1), \sigma(2), ...)$$

The values $\sigma(n)$ are called the coefficients of the power series or polynomial. Should σ be a polynomial, the largest $N \in \mathbb{N}_0$ such that $\sigma(N) \neq 0_R$ is called the *degree* of σ, and $\sigma(N)$ is called the *leading coefficient* of σ.

Whew! Here's our primary result:

Proposition. (3.25) If R is a ring, then $R[x]$, the collection of all polynomials over R, is a ring, called the *polynomial ring* over R. Additionally,

1. If R is commutative, then so is $R[x]$.
2. If R is an integral domain, then so is $R[x]$.
3. $R[x]$ contains a subring isomorphic to R.

Proof. The first order of business is to define the operations $\cdot$ and $+$ on $R[x]$. Here's some space for us to do this:

Now we've got a few conditions to check. As a refresher, it's probably not a bad idea to list the ring axioms here, given for a ring A:

1. $(A, +)$ is an abelian group with identity 0_R.
2.
3.
4.

Here's some room for you to verify each of these (roughly, at least!):

If we agree to write $x = (0, 1, 0, ...)$ to denote this particular element of $R[x]$, then we can do arithmetic using x as with any other polynomial.

In particular, we can compute...

1. $x \cdot \sigma$ for a given $\sigma = (s_0, s_1, s_2, ...)$:

2. x^n for any $n \in \mathbb{N}_0$:

How can we use x to rewrite a given polynomial $\sigma = (s_0, s_1, s_2, ...)$ in a form we're more familiar with?

Since we're used to thinking of polynomials as functions, it's nice to know that we can still get away with this:

Definition. If $\sigma \in R[x]$ is a polynomial, it defines a function $\sigma : R \to R$ by *evaluation*: if $\sigma(x) = s_0 + s_1 x + \cdots + s_n x^n$, then

$$\sigma(r) = s_0 + s_1 r + \cdots + s_n r^n.$$

Finally, let's note that in case R is a field we can talk about rational functions in much the same way we're used to doing so:

Definition. If k is a field, then the field of fractions of $k[x]$ (denoted $k(x)$) is called the *function field over* k, consisting of the *rational functions over* k, of the form $\frac{f(x)}{g(x)}$ for polynomials f and g.

Once we've learned a bit more about ring homomorphisms, ideals, and quotient rings, we'll be able to say a bit more about the significance of polynomials. It's to homomorphisms that we'll now turn!

Homework from Section 3.3. Here are a few problems dealing with this section; committee problems are marked with "**(CP)**":

1. Problem 3.32 from page 242.
2. **(CP)** Problem 3.33 from page 242.
3. Let R be a ring and $n \in \mathbb{N}_0$. Let $R_n[x]$ be the collection of polynomials of degree at most n.
   (a) True or false: $R_n[x]$ is a ring.
   (b) **(CP)** How much structure can we get away with placing on $R_n[x]$? Prove your assertion.

The schedule for this homework is as follows:

- **Monday, January 25th.** Committee problems due.
- **Wednesday, January 27th.** The committees will present their reports.
- **Friday, January 29th.** Final drafts of all problems due.