Ring homomorphisms and related concepts, Part II

As we continue to investigate rings, we'd really like to follow the same course we plotted for groups, first defining quotient rings and then indicating some homomorphism theorems.

As we saw before, we can't define the quotient of a group by any old subgroup, it had to be ______________ . With rings, we can't define the quotient of a ring by a subring, but rather an ___________ :

Definition. Let R be a ring, and let $I \subseteq R$ such that

1. $0_R \in I$,
2. $a, b \in I \Rightarrow$ _____ $\in I$, and
3. $a \in I, r \in R \Rightarrow$ _____ $\in I$.

Then I is called a _________________ of the ring R. If in (3) we demand that _____ be in I for all $a \in I$ and $r \in R$, we call I a _________________ of R. If both of these conditions hold, I is simply called an ___________ of R.

So what?

Proposition (3.38). Let $\phi : R_1 \to R_2$ is a ring homomorphism, then $\ker(\phi)$ is an ideal of R.

Proof.

Examples. What are the kernels of each of the exemplary homomorphisms we defined in the last handout?

Example. Let R be a ring, and let $a_1, a_2, ..., a_n \in R$. Then the left ideal generated by the a_i (denoted $(a_1, ..., a_n)$ is given by

$$\{r_1 a_1 + r_2 a_2 + \cdots + r_n a_n \mid r_i \in R\}.$$

That is, we take the set of "left" linear combinations of the a_i.

Here's some space in which you can check this is really an ideal:

If $n = 1$ above (so that there's just a single a_i, a_1), we call the ideal $(a_1, ..., a_n) = (a_1)$ a ______________ *ideal.* It consists of the (left) multiples of a_1.

We should note that if R is ________________ then any one-sided ideal is a plain ol' ideal (there's no need to worry about on which side multiplication must take place).

Proposition. Let R be a ring, and let the ideal I contain the unit u. Then $I = R$.
Proof.

Proposition (3.43). A nonzero commutative ring R is a field if and only if its only ideals are $\{0_R\}$ and R.
Proof.

Proposition (3.44). A ring homomorphism $\phi : R_1 \to R_2$ is an injection if and only if $\ker(\phi) = \{0_{R_1}\}$.
Proof.

Corollary (3.45). If $\phi : k \to R$ is a homomorphism from the field k to the ring $R \neq \{0_R\}$, then k is injective.

Proof.

Okay, here are a couple more exercises due on *Friday, January 30th* at 5:00 p.m., just to tide you over (neither are committee problems):

1. Let $\phi : k \to R$ be a ring homomorphism from the field k to the nonzero ring R. Assume that ϕ is surjective. Prove that R is a field. Can you say even *more* about R?

2. Show that there exists a surjective ring homomorphism $\phi : R \to k$ from a ring R to a field k such that R is *not* a field. (*Hint*: we don't know all that many examples of fields yet, so don't look too far!)