Extending ourselves

Consider the following types of extensions over $\mathbb{Q}$:

1. $\mathcal{E}$ is the set of all extensions of $\mathbb{Q}$.
2. $\mathcal{N}$ is the set of all normal extensions of $\mathbb{Q}$.
3. $\mathcal{A}$ is the set of all algebraic extensions of $\mathbb{Q}$.
4. $\mathcal{F}$ is the set of all finite extensions of $\mathbb{Q}$.
5. $\mathcal{R}$ is the set of all radical extensions of $\mathbb{Q}$.
6. $\mathcal{P}$ is the set of all pure extensions of $\mathbb{Q}$.

Proposition. $\mathcal{P} \subseteq \mathcal{R} \subseteq \mathcal{F} = \mathcal{A} \subseteq \mathcal{E}$, $\mathcal{N} \subseteq \mathcal{E}$, and there are examples of fields in both $\mathcal{N} \cap \mathcal{X}$ and $\mathcal{N} \cap \mathcal{X}^c$ for $\mathcal{X} \in \{\mathcal{F}, \mathcal{R}, \mathcal{P}\}$.

Proof. First note that $\mathcal{F} = \mathcal{A}$ from an earlier proposition, so we will only deal with $\mathcal{F}$ from here on.

We prove the existence of a field in each of the regions necessary to establish the truth of the proposition:

1. $\mathbb{Q} \in \mathcal{P} \cap \mathcal{N}$. Obviously $\mathbb{Q} = \mathbb{Q}(1)$ a pure extension of $\mathbb{Q}$ itself, of type $m = 1$. Since $\mathbb{Q}$ is the splitting field of the irreducible polynomial $x - 1 \in \mathbb{Q}[x]$, $\mathbb{Q}$ is normal over itself.
2. $\mathbb{Q}(\sqrt[3]{2}) \in \mathcal{P} \setminus \mathcal{N}$. The field $\mathbb{Q}(\sqrt[3]{2})$ is a pure extension of $\mathbb{Q}$ of type 3, since $(\sqrt[3]{2})^3 = 2 \in \mathbb{Q}$. But the irreducible polynomial $x^3 - 2 \in \mathbb{Q}[x]$ of which $\sqrt[3]{2}$ is a root does not split over $\mathbb{Q}(\sqrt[3]{2})$, so this is not a normal extension of $\mathbb{Q}$.
3. $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \in (\mathcal{R} \cap \mathcal{N}) \setminus \mathcal{P}$. Since $\mathbb{Q} \le \mathbb{Q}(\sqrt{2}) \le \mathbb{Q}(\sqrt{2}, \sqrt{3})$ witnesses a chain of pure extensions of type 2, by definition $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a radical extension of $\mathbb{Q}$. It is normal because it is also the splitting field of the quartic polynomial $x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3)$ over $\mathbb{Q}$. However, since the field in question cannot be obtained from $\mathbb{Q}$ by adjoining a single u such that $u^k \in \mathbb{Q}$ (why not?), the extension is not pure.
4. $\mathbb{Q}(\sqrt[3]{5 + \sqrt{2}}) \in \mathcal{R} \setminus (\mathcal{N} \cup \mathcal{P})$. The number $\sqrt[3]{5 + \sqrt{2}}$ is a root of the irreducible sextic polynomial $x^6 - 10x^3 + 23$, but this polynomial does not split in the given extension, meaning that $\mathbb{Q}(\sqrt[3]{5 + \sqrt{2}})$ is not normal. Note that $(\sqrt[3]{5 + \sqrt{2}})^3 \in \mathbb{Q}(\sqrt{2})$, however, so that $\mathbb{Q} \le \mathbb{Q}(\sqrt{2}) \le \mathbb{Q}(\sqrt{2}, \sqrt[3]{5 + \sqrt{2}})$ is a radical tower over extensions showing that our extension is radical. (Why is it not pure?)

5. K_1 = the splitting field of any unsolvable (by radicals) quintic polynomial $f(x) \in \mathbb{Q}[x]$ lies in $(\mathcal{F} \cap \mathcal{N}) \setminus \mathcal{R}$. Since K_1 is the splitting field of some polynomial and is therefore clearly a finite extension, it is thus normal (by Theorem 5.29). However, it is not a radical extension (and is therefore not a pure extension) of $\mathbb{Q}$, by hypothesis. (**Note:** technically we've not shown such extensions exist, but they follow as a consequence of Galois theory.)

6. $K_2 = K_1(\sqrt[3]{n})$, where n is any integer such that $x^3 - n$ is irreducible over K_1, lies in $\mathcal{F} \setminus (\mathcal{N} \cup \mathcal{R})$. Clearly this extension lies in some splitting field of finite degree over $\mathbb{Q}$, and is therefore itself of finite degree over $\mathbb{Q}$. However, since the polynomial $f(x)(x^3 - n)$ does not split over K_2 but the root $\sqrt[3]{2}$ lies in K_2, the extension is not normal. Moreover, since K_1 is not radical, K_2 is not radical either.

7. $\mathbb{C} \in (\mathcal{E} \cap \mathcal{N}) \setminus \mathcal{F}$. Since $\mathbb{C}$ contains all roots of all polynomials over $\mathbb{Q}$, $\mathbb{C}$ is a normal extension of $\mathbb{Q}$, as any irreducible $f(x) \in \mathbb{Q}[x]$ with a root in $\mathbb{C}$ (that is, *any* irreducible) splits in $\mathbb{C}$. However, were $\mathbb{C}$ a finite extension of $\mathbb{Q}$, every one of its elements would be algebraic over $\mathbb{Q}$. However, a deep classical theorem says that $\pi \in \mathbb{C}$ is not algebraic, so $\mathbb{C}$ cannot be a finite extension of $\mathbb{Q}$.

8. $\mathbb{R} \in \mathcal{E} \setminus (\mathcal{N} \cup \mathcal{F})$. Since $\pi \in \mathbb{R}$ is not algebraic, $\mathbb{R}$ cannot be a finite extension of $\mathbb{Q}$. Moreover, $\mathbb{R} \notin \mathcal{N}$ since $\mathbb{R}$ contains the root $\sqrt[3]{2}$ of the polynomial $x^3 - 2$, which does not split over $\mathbb{R}$.

$\diamondsuit$