MATH 462, Section 1
Abstract Algebra II
Spring 2009

Handout 13: Gimme an 'F'! Gimme an 'H'! Gimme a 'T'!

We're almost ready to present the ring-theoretic analogue of a much-beloved group-theoretic standard. (And in the process we'll construct a number of two-word hyphenated phrases.)

Definition/Proposition. Let R be a ring and $I \subseteq R$ an ideal. Then the function $\pi : R \to R/I$ defined by $\pi(r) = r + I$ is a surjective homomorphism called the ______________ *map.* (Often it's denote π for "projection," though just as often you'll see ν for " ______________ .")

So what?

Proposition (3.108). Let R be a ring. If I is an ideal in R, then I is the kernel of some homomorphism from R to another ring, S, and $S \cong R/I$.

Proof. This is one of those "follow-your-nose" deals:

First Homomorphism Theorem (ring version). Let R and S be rings, and let $\phi : R \to S$ be a homomorphism. Then $R/\ker(\phi) \cong \operatorname{im}(\phi)$.

Proof. First, we not define the isomorphism (your textbook denotes it $\tilde{\phi}$) that's going to do the trick:

Next, prove that it works (note: you've got to prove that the map is *well-defined* as well!):

So, what's the commutative diagram associated with this theorem? Here's some space for a quick sketch:

Okay, now let's go crazy with some

Examples. In the space below, let's see how many concrete instances of the FHT we can come up with, using the familiar rings, ideals, and homomorphisms with which we're already familiar:

Homework. For class on Friday, decide for yourself which of the following two courses you'd like to pursue next:

1. **Continue with quotients immediately.** Our first option continues in Section 3.8 of your text without interruption: we'll go straight into a deeper consideration of the structure of quotient rings and finite fields. From there, it's on to a little bit of "general" linear algebra, before returning to Galois theory and, time permitting, other topics.

2. **Universal algebra.** The second option would be to go "rogue" for a little while and spend a few days talking about what is in some ways the most abstract branch of abstract algebra, *universal algebra*. This topic addresses what you get when you generalize the notions of "operation," "substructure," and "homomorphism." If we follow this path, we would *afterward* return to the book and pick up where we left off.

Let me know what you think!