MATH 462, Section 1
Abstract Algebra II
Spring 2009

Exam 3

This take-home exam is worth a total of 50 points; point values are given with each question below. Please answer all questions clearly and completely on your own paper. You may use your notes and textbooks while completing this exam, but you may not consult with your colleagues, the folks in the Math Lab, or any faculty besides me. You may ask me questions, but I promise to not be very helpful.

Your exam is due by 5:00 p.m. on *Monday, May 4th.*

1. (10 points total; 5 point each)

   (a) Prove only the Sum Rule and the Product Rule in Problem 3.36 on page 242 of your text. (You can assume the truth of the other rules as needed in part (b) below.)

   (b) Do Problem 3.37 on page 243 of your text.

2. (25 points total; 5 points each) In universal algebra terms, a *semigroup* S is an algebra of type $\mathcal{F} = \mathcal{F}_2 = \{\cdot\}$, where $\cdot$ satisfies
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
   forall $a, b, c \in S$. That is, we only demand that S has an associative binary product, which we often write by "juxtaposition" instead of $\cdot$. *I.e.*, ab means the same thing as $a \cdot b$.

   (a) Let S be any nonempty set and define $\cdot$ by $a \cdot b = a$ for all $a, b \in S$. Prove that $(S, \cdot)$ is a semigroup. (Such a semigroup is called a *left zero semigroup.*)

   (b) Prove that any two left zero semigroups of the same cardinality are isomorphic. (*Hint:* let $\phi : S_1 \to S_2$ be *any* bijection...)

   (c) Let $\phi : S_1 \to S_2$ be a semigroup homomorphism. Show that if S_1 is a left zero semigroup, then so is S_2.

   (d) Let S be a left zero semigroup, and let $\theta \in \mathrm{Eq}(S)$ be *any* equivalence relation on S. Prove that θ is a congruence. (Thus $\mathrm{Con}(S) = \mathrm{Eq}(S)$ in this case, whereas in general only "$\subseteq$" holds.)

   (e) Prove that if $m \leq n$ are natural numbers and S_m and S_n are left zero semigroups of cardinality m and n, then $S_m \cong S_n/\theta$ for some congruence θ. Thus, any left zero semigroup maps homomorphically onto any smaller left zero semigroup, so the theory of these semigroups is not particularly interesting! (*Hint:* by (b) and (c) you really just have to show that you can find an congruence with the right number of equivalence classes...by (d) you can reduce this to looking only at equivalence relations. Put this together clearly!)

3. (10 points) Do Problem 5.29 on page 471 of your text.

4. (5 points) Let R be a ring, and recall the definition of *fraction field* $F = \mathrm{Frac}(R)$ of R, given on page 233 of your text. Prove using the definition that if $R \cong R'$, then $\mathrm{Frac}(R) \cong \mathrm{Frac}(R')$.