MATH 462, Section 1
Abstract Algebra II
Spring 2009

Exam 2

This take-home exam is worth a total of 50 points; point values are given with each question below. Please answer all questions clearly and completely on your own paper. You may use your notes and textbooks while completing this exam, but you may not consult with your colleagues, the folks in the Math Lab, or any faculty besides me. You may ask me questions, but I promise to not be very helpful.

Your *initial draft* of the exam is due on *Monday, April 6th* at 5:00 p.m.; it will then be returned to you and you will have a few days to perform revisions for up to 1/3 credit back. Revisions will be due at 5:00 p.m. on *Friday, April 10th.*

1. (5 points) Find all irreducible quadratic polynomials in $\mathbb{Z}_5[x]$.
2. (5 points) Find three irreducible 4th degree polynomials in $\mathbb{Z}_5[x]$.
3. (5 points) Find a field k of order 81, and *prove* that your choice of k is a field.
4. (5 points) Find a field k of order 961, and *prove* that your choice of k is a field.
5. (10 points) Let $\mathbb{Q}(\sqrt{5})$ denote the set

   $$\{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}.$$

   Prove that $\mathbb{Q}(\sqrt{5})$ is a field by proving that it's isomorphic to a certain quotient ring and appealing to a certain theorem from your textbook. (Of course, you can show that $\mathbb{Q}(\sqrt{5})$ is a field directly, but where's the fun in that?)
6. (20 points total; 5 points each) We've already talked about the universal algebraic concepts of *homomorphism* and *subalgebra*. A third way of constructing new algebras from old ones is by using *products*. Here we'll look at this construction a little bit.

   (a) Given $\mathcal{F}$-algebras A and B, we define the *product* $A \times B$ by letting the underlying set be

   $$\{(a, b) \mid a \in A \text{ and } b \in B\}.$$

   For a given n-ary operation f, describe how to define $f^{A\times B}$ so that $A \times B$ becomes an $\mathcal{F}$-algebra. (*Hint:* Note you must explain what ordered pair $f^{A\times B}\big((a_1, b_1), (a_2, b_2), ..., (a_n, b_n)\big)$ must be.)

   (b) If A and B are $\mathcal{F}$-algebras, prove that $A \times B \cong B \times A$ by finding an explicit isomorphism from one to the other and proving that it's an isomorphism.

   (c) Let A and B be $\mathcal{F}$-algebras. Define the relation θ_A on $A \times B$ as follows:

   $$((a, b), (a', b')) \in \theta_A \Leftrightarrow a = a'.$$

   That is, two pairs are equivalent if they're equal on the first coordinate. Prove that this relation is a congruence on $A \times B$.

   (d) Let A and B be $\mathcal{F}$-algebras. Define the function $\phi : A \times B \to A$ by $\phi\big((a, b)\big) = a$. Prove that $\ker(\phi) = \theta_A$ and that $\operatorname{im}(\phi) = A$. What does the First Homomorphism Theorem then tell you?