MATH 462, Section 1
Abstract Algebra II
Spring 2009

Exam 1

This take-home exam is worth a total of 50 points; each question is worth 10 points. Please answer all questions clearly and completely on your own paper. You may use your notes and textbooks while completing this exam, but you may not consult with your colleagues, the folks in the Math Lab, or any faculty besides me. You may ask me questions, but I promise to not be very helpful.

Your *initial draft* of the exam is due on *Friday, February 20th* at 5:00 p.m.; it will then be returned to you and you will have a few days to perform revisions for up to 1/3 credit back.

1. (5 points each) Let R and S be commutative rings and let $\phi : R \to S$ be a surjective ring homomorphism.
   (a) *True or False* (please justify your answer with a proof or a counterexample, as appropriate): if R is a principal ideal domain, then so is S.
   (b) *True or False* (same deal!): if S is a principal ideal domain, then so is R.
2. Draw a single Venn diagram illustrating the containment relationships between all of the following sets: rings, commutative rings, integral domains, principal ideal domains, fields. For every pair A, B for which you claim $A \not\subseteq B$, justify your claim by showing that there is a ring R such that $R \in A$ but $R \notin B$.
3. Let R be a commutative ring and let I be an ideal in R. The *radical* of I, denoted by $\sqrt{I}$, is the set of all elements $r \in R$ such that $r^n \in I$ for some integer $n > 0$. (Here r^n is simply $r \cdot r \cdots r$, with n terms in the product.) Prove that $\sqrt{I}$ is also an ideal in R.
4. (5 points each) We say that the function $f : \mathbb{R} \to \mathbb{R}$ is *bounded* if there is some $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Define addition and multiplication of functions pointwise:
$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (f \cdot g)(x) = f(x)g(x)$$
   for all functions f and g. Let F be the set of bounded functions, and let F_D be the set of bounded *differentiable* functions.
   (a) Prove that F is a commutative ring, but it is not an integral domain.
   (b) Prove that F_D is a subring of F but it is not an ideal in F.
5. Let X be a given set. Recall that we can make the powerset $\mathcal{P}(X)$ into a ring by defining addition and multiplication as follows for all $A, B \subseteq X$:
$$A + B = (A \setminus B) \cup (B \setminus A) \qquad \text{and} \qquad A \cdot B = A \cap B.$$
   (The first of these formulas is just the symmetric difference of A and B.)
   (a) (3 points) Show that $(\mathcal{P}(X), +, \cdot)$ is *not* an integral domain.
   (b) (3 points) Describe the principal ideal (A) generated by a given $A \subseteq X$.
   (c) (4 points) Describe what it means for $A|B$ to hold in $\mathcal{P}(X)$. Use this to describe how to find the GCD of two sets $A, B \subseteq X$.