# *Tools of the Trade: Notation and Definitions, volume V*

More! This time, the focus is on relations and functions.

(1) Recall that a *relation $R$ between* the sets $S$ and $T$ is nothing more than a subset of $S \times T$. That is, $R \subseteq S \times T$. Notice that a relation is therefore a set of pairs, the first coordinate of which lies in $S$, and the second coordinate of which lies in $T$. (As we saw in one of the homework problems, there are $|\mathcal{P}(S \times T)| = 2^{|S \times T|} = 2^{|S|\cdot|T|}$ possible relations between $S$ and $T$.)

A *relation $R$ on* a set $S$ is merely a relation between $S$ and itself. That is, a relation on $S$ is a subset of $S \times S$.

Given any relation $R \subseteq S \times T$, the *inverse* of $R$, denoted $R^{-1}$, is the set

$$\{(t, s) \mid (s, t) \in R\}.$$

Clearly this is a relation between $T$ and $S$.

(2) An *equivalence relation* on $S$ is a relation $R \subseteq S \times S$ that satisfies three properties:
(a) *Reflexivity*: $(\forall s \in S)\ (s, s) \in R$.
(b) *Symmetry*: $(\forall s, t \in S)\ (s, t) \in R \Rightarrow (t, s) \in R$.
(c) *Transitivity*: $(\forall s, t, u \in S)\ (s, t) \in R$ and $(t, u) \in R \Rightarrow (s, u) \in R$.

If $R \subseteq S \times S$ is an equivalence relation and $(s_1, s_2) \in S$, we often write $s_1 \sim_R s_2$ instead. Note that an equivalence relation generalizes the notion of "equal to," since ordinary equality is an equivalence relation.

Finally, recall that if $R \subseteq S \times S$ is an equivalence relation on $S$, then the set $[s]_R = \{s' \in S \mid (s, s) \in R\}$ is called the *equivalence class of $s'$, modulo $R$*. The set $S$ is the disjoint union of the equivalence classes modulo $R$; that is, the set of these classes forms a partition of $S$.

**Example.** Let $k \in \mathbb{N}$. Then define the relation $\equiv_k$ by $(m, n) \in \equiv_k$ if and only if $k$ divides $m - n$ (this is written $k|(m - n)$). Generally we write $m \equiv_k n$ or $m \equiv n \mod k$ and say $m$ and $n$ are *equivalent modulo $k$*. This is an equivalence relation with $k$ equivalence classes.

(3) Just like equivalence relations generalize "=," *order relations* generalize "≤." We say that $R \subseteq S \times S$ is an order relation on $S$ if it is (a) reflexive, (b) transitive, and (c) *antisymmetric*:

$$(s, t) \in R \text{ and } (t, s) \in R \Rightarrow s = t.$$

If $R$ is an order relation on $S$ and $(s, t) \in R$, we often write $s \leq_R t$ instead.

If $S$ is a set on which an order relation $R$ is defined, we refer to the pair $(S, \leq_R)$ a *partially ordered set*, or *poset*. *Total* (or *linear*) *orders* are special kinds of order relations: $R$ is a total order on $S$ if for all $s, t \in S$, either $s \leq_R t$ or $t \leq_R s$ holds. A total order is called a *wellordering* of $S$ if every nonempty subset of $S$ has a least element with respect to the order $R$. For example, while $\mathbb{Z}$ is a totally ordered set with respect to the usual order, it is not a wellordered set. On the other hand, $\mathbb{N}$ *is* wellordered.

(4) A *function* $f \subseteq S \times T$ is a relation between $S$ and $T$ such that for every $s \in S$ there is a *unique* $t \in T$ such that $(s, t) \in f$. We often denote this relationship by $f(s) = t$. $S$ is called the *domain* of the function $f$, and $T$ is called the *range*. The set

$$\{t \in T \mid (\exists s \in S)\ f(s) = t\}$$

is called the *image* of $f$. The image is the set of elements in $T$ "hit" by the function $f$.

Note that image$(f) \subseteq$ range$(f)$; if these sets are in fact equal, $f$ is said to be *onto*, or *surjective*. The function $f$ is called *one-to-one*, or *injective*, if

$$(\forall s_1, s_2 \in S)\ f(s_1) = f(s_2) \Rightarrow s_1 = s_2.$$

If $f$ is both surjective and injective, we say that it is a *bijection*. A bijection from $S$ to $T$ is essentially just a "renaming" of the elements of $S$.

Since functions are just special cases of relations, we can define the *inverse*, $f^{-1}$, of a function $f$ just as we did the inverse of a relation. (Recall that in general that the inverse of a function is not necessarily a function anymore!) If $f$ is injective, $f^{-1}$ *is* a function. Moreover,

$$\text{domain}(f^{-1}) = \text{image}(f) \text{ and range}(f^{-1}) = \text{domain}(f).$$

Note that the inverse of a bijection is also a bijection.

Finally, two functions $f$ and $g$ are said to be *equal* if they're equal as sets. For this to be true, both the domains and the images must be equal, and the "matchings" defined by $f$ and $g$ must agree.