Tools of the Trade: Notation and Definitions, volume III

Presenting another installment in a long line of useful note sets on notation and definitions!

(1) **A Review of Mathematical Induction.** There are three steps to an inductive proof of a statement $P(n)$ made about the natural numbers (or perhaps some subset of $\mathbb{N}$, like $\{n \in \mathbb{N} \mid n \geq N\}$, N some fixed natural number):

- (a) **Base Case:** prove that the statement $P(n)$ is true for the case $n = 1$ (or for the case $n = N$, if the smallest value for which $P(n)$ is claimed true is not 1).
- (b) **Inductive Hypothesis:** assume that the statement $P(n)$ is true for some fixed but arbitrary value of n. (In the case of *Strong Induction*, we assume that $P(k)$ is true for all k up to and including n.)
- (c) **Inductive Step:** by assuming the inductive hypothesis, and using whatever other mathematical formulas and rules you need, prove that $P(n + 1)$ is true.

(2) **Set Theory notation.** You should be familiar with the "curly bracket" notation for sets: placing "{" and "}" around a collection of objects makes it into a set. Also, you should be familiar with the following notations:

- (a) *element inclusion* $\in$: $a \in A$ indicates that a is an element of the set A.
- (b) *set containment* $\subseteq$ and *proper subset containment* $\subset$: $A \subseteq B$ indicates that A is a subset of B, and $A \subset B$ indicates that A is a *proper* subset of B. (Recall what these definitions mean!)
- (c) *set specification*: by writing $\{s \in S \mid P(s)\}$ we create a set whose elements are all those lying in the set S which also satisfy the property P. For example,

$$\{s \in \mathbb{Z} \mid 2|s\}$$

gives the set of even integers, while

$$\{n \in \mathbb{N} \mid 31 \leq n \leq 107\}$$

gives all natural numbers between 31 and 107, including these values.
- (d) *powerset* $\mathcal{P}(S)$: the set $\mathcal{P}(S)$ is the powerset of the set S, the set whose *elements* are the subsets of S.
- (e) *union* $\cup$ and *intersection* $\cap$: $A \cup B$ is the collection of elements lying in at least one of A or B; $A \cap B$ is the collection of elements lying in *both* A and B. Recall that we say A and B are *disjoint* sets if $A \cap B = \emptyset$.
- (f) If $A_1, A_2, ..., A_n$ are nonempty subsets of a set A such that
  - (i) $i \neq j \Rightarrow A_i \cap A_j = \emptyset$ and
  - (ii) $A_1 \cup A_2 \cup \cdots \cup A_n = A$,

  we say that the collection $\{A_1, A_2, ..., A_n\}$ forms a *partition* of A, and the subsets A_i are called the *classes* of the partition.

Our next episode of *Tools of the Trade* will continue with more set theory notation. Stay tuned!