Tools of the Trade: Notation (and Definitions!), volume II

Our special series of handouts continues with this one, a summary of important definitions and proof techniques!

(1) **Common proof techniques.** Let's say you need to prove the statement "If P, then Q" (that is, $P \Rightarrow Q$). You could prove this statement by...

(a) **Direct Proof:** Assume P, and from this derive Q by means of acceptable mathematical formulas and logical arguments.

(b) **Indirect Proof (a.k.a. Proof by Contradiction):** Assume that Q is false, and from this derive a contradiction to the hypotheses or to some fact from mathematics (like the fact that 0 is even!).

(c) **Proof by Contraposition:** Prove instead the logically equivalent statement $(\neg Q) \Rightarrow (\neg P)$, usually directly.

(2) **Useful definitions.** The following are a few useful definitions we've encountered so far.

(a) Let $a, b \in \mathbb{Z}$. We say that a *divides* b, written $a|b$, if there is an integer c satisfying $b = ac$. We could write this entirely in quantifiers as follows:

$$(\forall a, b \in \mathbb{Z}) \left[a|b \Leftrightarrow (\exists c \in \mathbb{Z})\ b = ac \right]$$

(b) A natural number $n \in \mathbb{N}$ is *prime* if $n \geq 2$ and its only positive divisors are itself and 1. If $n \geq 2$ is not prime, it is called *composite.* 1 is considered neither composite nor prime.

(c) Let $k \geq 1$ be an integer. A *perfect kth power* is any integer n having the form m^k for some other integer m. For example, $n = 64$ is a perfect third power (or a *perfect cube*), because $64 = 4^3$. (Here, $m = 4$ and $k = 3$.) If there is *some* k such that n is a perfect kth power, we can call n simply a perfect power.

Note: What natural numbers are perfect 1st powers?