MATH 280, Section 1
Foundations of Mathematics
Spring 2009

Relations, Relations, Relations!

I'm sure it wouldn't hurt any of us to take a moment to take stock of the sorts of relations we've studied in the past few weeks.

Let's just begin by reviewing the definitions...

Definitions. Let S and T be any sets. A *relation on S* is *any* subset R of $S \times S$. That is, $R \subseteq S \times S$, or $R \in \mathcal{P}(S \times S)$.
An *equivalence relation on S* is any relation R on S satisfying three properties:

(1) *Reflexivity.* $(\forall s \in S)(s, s) \in R$.
(2) *Symmetry.* $(\forall s, t \in S)(s, t) \in R \Rightarrow (t, s) \in R$.
(3) *Transitivity.* $(\forall s, t, u \in S)(s, t), (t, u) \in R \Rightarrow (s, u) \in R$.

Equivalence relations generalize the notion of "equal to": if you ever have trouble understanding an equivalence relation, just think of it in terms of "=."

An *order relation on S* is any relation R on S satisfying both reflexivity and transitivity, and instead of symmetry,

(2′) *Antisymmetry.* $(\forall s, t \in S)(s, t), (t, s) \in S \Rightarrow s = t$.

Order relations generalize the notion of "less than or equal to": if you ever have trouble understanding an order relation, just think of it in terms of "$\leq$."

Finally, we say that a relation R between S and T (remember that in this case $R \subseteq S \times T$!) is a *function* from S to T if

$$(\forall s \in S)(\exists!\ t \in T)(s, t) \in R.$$

The notation "$\exists!$" means "there exists a unique."

We note that relations are ***very*** general, while the special kinds of relations we discussed above are quite special and don't arise very often. That is,

most relations are not equivalence relations, order relations, *or* functions.

Examples. Let $S = \{x, y, z\}$.

(1) $R_1 = \{(x, x), (y, y), (z, z), (x, y), (y, x)\}$ is an equivalence relation on S that is neither an order relation on S nor a function from S to S.
(2) $R_2 = \{(x, x), (y, y), (z, z), (x, z)\}$ is an order relation on S that is neither an equivalence relation on S nor a function from S to S.
(3) $R_3 = \{(x, y), (y, z), (z, z)\}$ is a function from S to S that is neither an equivalence relation on S nor an order relation on S. Note that R_3 is not a *bijection* (that is, a one-to-one and onto function), because $R_3(y) = R_3(z) = z$ means that R_3 is not injective.

On the other hand, there are relations that are special indeed: the *trivial* relation on S, let's call it I, defined by

$$I = \{(s, s) \mid s \in S\}$$

is an equivalence relation on S, an order relation on S, and a bijection from S to itself, all at once!

One way to get a better handle on these different kinds of relations is to count them: if we see just how many relations there are of a certain type, we can better appreciate how rare they are.

Enumerating relations. Let's take finite sets S with n elements and T with m elements. That is, $|S| = n$ and $|T| = m$, where $m, n < \infty$.

(1) **Relations.** Since $|S \times S| = |S| \cdot |S| = n^2$, and since for any set X, $|\mathcal{P}(X)| = 2^{|X|}$, we know there are

$$|\mathcal{P}(S \times S)| = 2^{|S \times S|} = 2^{n^2}$$

relations on S. Wow! For example, if $|S| = 3$, there are $2^9 = 512$ relations on S.

(2) **Equivalence relations and order relations.** These turn out to be quite difficult to count, as there's no "closed form" for the number of these relations on a set of a given size. For instance, if $|S| = 3$, there are 5 equivalence relations and 19 order relations on S.

(3) **Functions.** It's not too hard to see that the number of functions from S to T is given by $|T|^{|S|} = m^n$. Why? For each of the n elements in S, we have $|T| = m$ choices for which element we "send it to" under our function.

In the special case $S = T$, we therefore have n^n functions from S to itself. When $|S| = 3$, we get 27 functions from S to itself.

(4) **Bijections.** We counted the bijections from S to itself in class the other day. Indeed, if $S = \{s_1, s_2, ..., s_n\}$, we can describe our bijection by first saying which element s_1 goes to (n choices), then which element s_2 goes to ($n - 1$ choices), then s_3 ($n - 2$ choices), and so on, where s_{n-1} goes to one of 2 elements, and s_n's destination is then forced. Thus we obtain

$$n(n-1)(n-2) \cdots 2 \cdot 1 = n!$$

choices for our bijection.

For example, when $|S| = 3$, we get $3! = 6$ bijections from S to itself, as we saw in class.

These numbers really do make us appreciate the rarity of the special kinds of relations.

For example, if we look at the ratio of the number of bijections on a set S to the number of functions from S to itself, we obtain

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \cdots \cdot \frac{n-1}{n} \cdot \frac{n}{n} \leq \frac{1}{n},$$

which goes to 0 as $n \to \infty$. That is, most functions are not bijections.

Similarly, the ratio of the number of functions from a set S to itself to the number of relations on the set S is

$$\frac{n^n}{2^{n^2}} = \left(\frac{n}{2^n}\right)^n,$$

which you can show goes to 0 as $n \to \infty$.

The moral of the story is that relations are very, very basic creatures, and there are *many* of them. Relations of a *special* nature are much rarer, though, and that's why it's nice when we've got one!