MATH 280, Section 1
Foundations of Mathematics
Spring 2009

Relations, Part the Second

As promised at the end of the last worksheet, let's begin our discussion of *order relations*, another important sort of relation on a set S.

Example. Consider the set $S = \{1, 2, 3, 4, 5\}$, and the relation $R \subseteq S \times S$ on S defined by

$$R = \{(s, t) \mid s, t \in S, s \leq t\}.$$

Write out this relation, as a set, below:

As we'd mentioned on the last handout, this relation R essentially captures what is meant by "$\leq$": $(s, t) \in R$ if and only if $s \leq t$.

Here's another example in a similar vein:

Example. Define the relation R on the set $\mathcal{S} = \{A, B, \mathbb{N}, \mathbb{Z}, \mathbb{R}\}$ by

$$R = \{(S, T) \mid S, T \in \mathcal{S}, \text{ and } S \subseteq T\},$$

where $A = \{\pi, e\}$ and $B = \{-1, 0, 1\}$. Write out this relation explicitly as a set:

In this case, as in the previous one, the elements of the set are "ordered": there is some means of determining when one element of the set is "bigger" (or "smaller") than another.

Definition. Let $R \subseteq S \times S$ be a relation on S. We call R an *order relation* if it is both reflexive and transitive (just like an equivalence relation), and instead of symmetry it satisfies the following condition:

Antisymmetry. If $(s,t) \in R$ and $(t,s) \in R$ both hold, then $s = t$.

This condition generalizes the following fact about real numbers:

$$(\forall x, y \in \mathbb{R})\Big((x \leq y) \wedge (y \leq x) \Rightarrow x = y\Big).$$

Just like we often write $s \sim_R t$ for $(s,t) \in R$ if R is an equivalence relation, we can write $s \leq_R t$ whenever $(s,t) \in R$ if R is an order relation. (In fact, sometimes mathematicians will just write R using the symbol $\leq$ instead!)

The study of order relations is an incredibly deep and beautiful one, and we'll only have a chance to brush the surface. Let's start off with another

Definition. If P is a set on which the order relation R is defined, we call the pair $(P, \leq_R)$ a *partially ordered set* (or, more briefly, a *poset*).

Examples. Each of $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$, and $(\mathbb{R}, \leq)$ is a poset, where $\leq$ is defined as usual. If S is any set, the powerset $\mathcal{P}(S)$ gives us a poset $(\mathcal{P}(S), \subseteq)$, where $\subseteq$ is set containment, defined as usual.

Coming up with order relations can sometimes be a chore, but there are some that are pretty natural.

Exercise. Let $(P, \leq)$ be a poset, where $\leq$ is given by the order relation R. Define the *inverse* R^{-1} of R by

$$(s,t) \in R^{-1} \iff (t,s) \in R.$$

Prove that if R is an order relation, then so is R^{-1}.

Recall this means checking three properties:

Reflexivity. Let $s \in P$. We must show that $(r,r) \in R^{-1}$...

Antisymmetry. Let (s,t) and (t,s) *both* be in R^{-1}. We must show that $s = t$. Well, if both (s,t) and (t,s) are in R^{-1}, where else do they both lie?...

Transitivity. Let $(s,t), (t,u) \in R^{-1}$. We must show that $(s,u) \in R^{-1}$. Well, ...

The order $\leq_{R^{-1}}$ defined by the inverse of an order relation R is said to be *dual* to the original order. In this dual order, everything that was "small" is now "big," and *vice versa*.

Exercise. Let S be a set. Prove that the *trivial order* $R \subseteq S \times S$ defined by

$$R = \{(s, s) \mid s \in S\}$$

is an order relation. What is bigger than what, according to this order?

In mathematics it's often useful to do away with the "=" option allowed by the statement "$s \leq t$." That is, we might want to assert that s is *strictly less than* t: $s < t$.

Definition. If P is a poset under the order relation R, then we define a new relation, R_s, as follows:

$$(s, t) \in R_s \Leftrightarrow \Big((s, t) \in R \text{ and } s \neq t\Big).$$

Written in terms of "$\leq$" and "$<$," this is really saying nothing more than $s < t$ if and only if $s \leq t$ and $s \neq t$ both hold!

Note. This is *not* an order relation! Specifically, which conditions does it fail?

Let's discuss one more sort of order relation, and finally a special case.

Definition. Let S be a set, and let R be an order relation on S. R is called a *total ordering* on S if the following is true:

$$(\forall s, t \in S) \ \Big((s, t) \in R \vee (t, s) \in R\Big).$$

That is, either $s \leq_R t$ or $t \leq_R s$ is true, for *every* pair of elements in S.

Example. Which of the order relations we've considered today are total orderings?

Definition. A total ordering R on a set S is called a *wellordering* if it has the following property (called the *wellordering property*):

Every nonempty subset of S has a least element, modulo $\leq_R$.

Examples. Which of the total orderings we've considered are wellorderings?