Relations, Part the First

Let's take a brief trip back to the carefree days of Calc I, when you first encountered an analysis of the equation $x^2 + y^2 = 1$.

One of the first things pointed out to you at that time was that the points satisfying this equation *are not the graph of a function*! In fact, there are *two* functions at work here, if we solve for y in terms of x: $y = \pm\sqrt{1 - x^2}$.

However, the x- and y-values of those points *are* related somehow, in a manner given by that equation. In fact, we may call the set of points on the curve defined by $x^2 + y^2 = 1$ a *relation*.

Note. We will define functions soon, as special cases of relations; this is the opposite order in which these notions are defined in calculus, where you learn to differentiate functions before you can differentiate more general relations implicitly!

Definition. Given two sets S and T, a *relation* between S and T is a subset $R \subseteq S \times T$.

Notice that the graph of a function $f : S \to T$ is indeed a relation between S and T, if we think of the graph as a collection of points satisfying the function's equation:

$$\text{graph}(f) = \{(s, f(s)) \mid s \in S\} \subseteq S \times T.$$

However, the idea of a relation is much more general.

Returning to our first example, we note that the collection

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

is a relation between $\mathbb{R}$ and itself. (A relation between a set S and itself is called a relation *on* S.)

Examples. Consider the set $S = \{0, 1, 2, 3, 4, 5\}$. Write each of the indicated relations by listing the relations explicitly as sets.

(1) $R_1 = \{(s_1, s_2) \mid s_1 \text{ and } s_2 \text{ have the same parity}\}$.

(2) $R_2 = \{(s_1, s_2) \mid s_1 = s_2\}$.

(3) $R_3 = \{(s_1, s_2) \mid s_1 \leq s_2\}$.

These last two examples illustrate two types of relations on which we will focus most of our attention. The second-to-last essentially describes what we mean by "=" in the set S, while the last describes what is meant by "≤."

That is, by listing all pairs of elements that are meant to be "equal" to each other, we essentially *define* what "=" must mean. Likewise, by listing all pairs of elements which are related in the "less than or equal to" fashion, we merely *define* what is meant by "≤."

But we've all seen other examples of "equality": two *sets* are equal if and only if they possess exactly the same elements; two *functions* f and g are equal if and only if they have the same domains and the same values: $f(x) = g(x)$ for all x in their common domain (more on this in a couple of weeks). (We'll also address "≤" more generally in a little while.)

Let's get general:

Definition. An *equivalence relation* R on a set S is a relation on S satisfying the following three conditions:

(1) (*reflexivity*) $(\forall s \in S)\ (s, s) \in R$,
(2) (*symmetry*) $(\forall s, t \in S)\ (s, t) \in R \Rightarrow (t, s) \in R$, and
(3) (*transitivity*) $(\forall s, t, u \in S)\ (s, t) \in R$ and $(t, u) \in R \Rightarrow (s, u) \in R$.

Roughly speaking, anytime $(s, t) \in R$, s and t are "equal." Often $(s, t) \in R$ is written $s \sim_R t$, read "s is equivalent to t, modulo R."

Examples. Check that each of the relations defined below is an equivalence relation.

(1) $R_1 \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $R = \{(x, y) \mid x$ and y have the same parity$\}$.

(2) $R_2 \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ defined by

$$R = \{\big((x,y),(x',y')\big) \mid \sqrt{x^2+y^2} = \sqrt{(x')^2+(y')^2}\}.$$

As you can see, it's often not too hard to prove that a given relation is an equivalence relation. Also, it's clear from the above examples that often "equality" can be construed quite loosely.

Let $R \subseteq S \times S$ be an equivalence relation on S, and let $s \in S$. The set

$$[s]_R = \{t \in S \mid (s,t) \in R\}$$

is called the *equivalence class* of the element s with respect to R. Roughly, it's the set of all elements of S "equal to" s with respect to the relation R.

Theorem. Let R be an equivalence relation on S. The set $\{[s]_R \mid s \in S\}$ of equivalence classes with respect to R forms a partition of S. (Recall that this means S is the *disjoint* union of R's equivalence classes.)

Proof. Two things must be shown: that every element of S lies in *some* class, and that no element lies in *two* classes. See if you can show this in the space below:

The set $\{[s]_R \mid s \in S\}$ of equivalence classes with respect to R is sometimes called the *quotient of S modulo R*, and is denoted S/R. We can think of the quotient as being formed by "collapsing to a point" any elements that are defined to be equivalent by R. In many applications S/R has an important structure all its own.

Examples. Return to the previous example.

(1) How many equivalence classes does R_1 have? Describe them.

(2) How many equivalence classes does R_2 have? Draw the graph of several of them on a set of axes below.

There's one more incredibly important type of equivalence relation we ought to talk about, on the set $\mathbb{Z}$ of integers.

Let k be any integer (usually we take $k \geq 1$. If $m, n \in \mathbb{Z}$, we say that m and n are *congruent modulo* k (written $m \equiv n \bmod k$, or $m \equiv_k n$) if k divides $m - n$. That is, $m - n = ks$ for some other integer s.

Here's some room for you to prove this is indeed an equivalence relation:

When is $m \equiv_k n$ true for $k = 1$? For $k = 2$? $k = 3$? How about $k = 0$? Can you couch your observations in terms of *remainders*?

In the next handout we'll turn our attention to the other very important sort of relation, *order relations.*