Making a Statement

In order to do any kind of mathematics, we need to be able to determine the *truth* or *falsity* of validly formed *mathematical statements*. Let's spend today investigating these ideas, introducing the notion of *quantifiers* along the way.

- A (*valid*) *mathematical statement* is any "grammatically correct" assertion to which we can assign one of the two *truth values* **TRUE** or **FALSE**.

  Examples. You might want to indicate the truth value of each statement beside it!
  (1) $1 + 1 = 4$
  (2) There is a number x greater than any real number.
  (3) π is a rational number.

  (4)

  (5)

  (6)

- Sometimes the truth value of a statement is incalculable without further specifying the values of quantities contained in the statement. For example, the statement "$x^2 > 1$" might be true or it might not be, depending on what value the *variable* x has. There are two common ways of *quantifying* a variable in a statement:

  (1) The variable x in a mathematical statement is *universally quantified* in the statement "For all x in the set S, property $P(x)$ about the variable x holds." You should be able to identify S, $P(x)$, and the truth value of the statement, in each case below.

  Examples.
  (a) $x^2 \geq 0$ for all real numbers x.
  (b) For every integer x, $x < 97$.

  (c)

  (d)

(2) The variable x is *existentially quantified* in the statement "There exists x in the set S such that the property $P(x)$ holds."

Examples.

(a) For some rational number x, x is larger than π.

(b) There exists a circle in the Cartesian plane bounding an area larger than 1 square unit.

(c)

(d)

The set S of allowed values for a variable x is called its *universe*.

- Quantifiers arise so often that we seek special notation for them:
  - Universal quantifiers are written $(\forall x \in S)\ P(x)$. You should translate the first two universal statements above using this notation:

    (1)

    (2)

  - Existential quantifiers are written $(\exists x \in S)\ P(x)$. Translate the first two existential statements given above:

    (1)

    (2)

- Many statements are formed by combining quantifiers. When more than one quantifier appears, it's important to understand the *order of quantification*, as indicated in the following examples:

  Examples.

  (1) "For every real number a there exists a real number b such that $a < b$" means something other than...

  (2) ..."There exists a real number b such that for every real number a, $a < b$." (You should be able to translate these into our new notation and note the difference between the two!) Now write your own pair of statements where the order makes all of the difference:

  (3)

(4)

- New statements can be formed from old ones by *negation.* To negate a statement is to reverse its truth value. You should be able to negate the following statements, and then invent a couple of your own to negate:

  (1) There is a real number x such that $x + 6 = 7$.

  (2) For every integer z, $z > 0$.

  (3)

  (4)

  Below, translate each of the above statements, and their negations, into our notation for quantifiers (I've done the second one for you):

  (1)

  (2) $(\forall z \in \mathbb{Z})\ z > 0$; **negation:** $(\exists z \in \mathbb{Z})\ z \leq 0$. (Note that "not $z > 0$" is the same as "$z \leq 0$.")

  (3)

  (4)

  By the way, the negation of a statement $P(x)$ is generally written $\neg P(x)$. From the above examples, can you formulate a rule which allows you to negate a statement with quantifiers, replacing it with a new statement without a "$\neg$" out in front? Fill in your conjecture below:

  (1) $\neg\Big[(\forall x)\ P(x)\Big] =$

  (2) $\neg\Big[(\exists x)\ P(x)\Big] =$

  Example. Test your conjecture by negating the following statement using your rule, after translating it into our quantifier notation: "For every integer a there exists a real number b such that $\sqrt{b} > a$."