MATH 280, Section 1
Foundations of Mathematics
Spring 2009

Solutions to Homework 7

(1) (a) If $|S| = 1$, then there are $2^{|S\times S|} = 2^{1\cdot 1} = 2^1 = 2$ relations on S, but only 1 equivalence relation. In fact, if $S = \{s\}$, the relations are $\{(s,s)\}$ and $\emptyset$, only the first of which is an equivalence relation.

(b) If $|S| = 2$, then there are $2^{|S\times S|} = 2^{2\times 2} = 2^4 = 16$ relations on S, but only two of them are equivalence relations. In fact, if $S = \{s,t\}$, then the two equivalence relations are $\{(s,s),(t,t)\}$ and $S \times S = \{(s,s),(t,t),(s,t),(t,s)\}$. (Proving that these are the only two is straightforward; you really just have to point out that any other either fails reflexivity or symmetry.)

(2) Let D be the relation on $\mathbb{N} \cup \{0\}$ by $(m,n) \Leftrightarrow m|n \Leftrightarrow (\exists k \in \mathbb{Z})\ n = mk$.

(a) We must show that D is reflexive, antisymmetric, and transitive.

Reflexivity. Let $m \in \mathbb{N} \cup \{0\}$. Then since $m = m \cdot 1$, there is an integer k (namely, $k = 1$) such that $m = mk$. By definition, $m|m$, so $(m,m) \in D$, as needed.

Antisymmetry. Let $(m,n),(n,m) \in D$. Then since $m|n$ and $n|m$ both hold, there are integers $k, \ell \in \mathbb{Z}$ such that $n = mk$ and $m = n\ell$. Thus

$$n = mk = (n\ell)k = n(\ell k),$$

so $\ell k = 1$. Thus both k and ℓ must be 1, so finally $n = mk = m \cdot 1 = m$, as needed.

Transitivity. Let $(m,n),(n,p) \in D$. Then since $m|n$ and $n|p$ both hold, there are integers $k, \ell \in \mathbb{Z}$ such that $n = mk$ and $p = n\ell$. Thus

$$p = n\ell = (mk)\ell = m \cdot (k\ell),$$

and since $k\ell \in \mathbb{Z}$, by definition $m|p$, so $(m,p) \in D$, and we're done.

(b) We claim that $1 \leq_D x$ for all $x \in \mathbb{N} \cup \{0\}$. Indeed, $1 \leq_D x \Leftrightarrow (1,x) \in D \Leftrightarrow 1|x$. We must thus find $k \in \mathbb{Z}$ such that $x = 1 \cdot k$. But $k = x$ works, so we're done.
We also claim that $x \leq_D 0$ for all $x \in \mathbb{N} \cup \{0\}$. Indeed, $x \leq_D 0 \Leftrightarrow (x,0) \in D \Leftrightarrow x|0$. We must thus find $k \in \mathbb{Z}$ such that $0 = x \cdot k$. But $k = 0$ works, so we're done.

(3) As in the problem's statement, let S be a fixed and arbitrary set. (Note that we *cannot* select any particular set S!)

(a) The smallest equivalence relation on S (with respect to $\subseteq$) is $R_s = \{(s,s) \mid s \in S\}$. To see that this is an equivalence relation, note that it's obviously reflexive, that if $(s,t) \in R_s$ then $s = t$, so $(t,s) = (s,s) \in R_s$, and finally that if $(s,t),(t,u) \in R_s$, then $s = t = u$, so $s = u$ and $(s,u) \in R_s$. Moreover, if R is any other equivalence relation, it must be reflexive, and therefore must contain all pairs (s,s). Since these are the pairs that make up R_s, $R_s \subseteq R$, so R_s is indeed the smallest equivalence relation.

(b) The largest equivalence relation on S (with respect to $\subseteq$) is $R_\ell = S \times S$. You've already proven this is an equivalence relation (see Exam 2, Problem 2!). Note that it's trivially the largest equivalence relation since if R is any other equivalence relation on S, by *definition* $R \subseteq S \times S = R_\ell$, so we're done.

(4) (a) Let R_1 and R_2 both be equivalence relations on S, and consider $R = R_1 \cap R_2$. We check the usual conditions:

Reflexivity. Let $s \in S$. Then since R_1 and is equivalence relation, $(s,s) \in R_1$ by reflexivity of R_1. Similarly $(s,s) \in R_2$, and thus $(s,s) \in R_1 \cap R_2 = R$, as needed.

Symmetry. Let $(s,t) \in R$. Then by definition of intersection, $(s,t) \in R_1$ and $(s,t) \in R_2$. Thus by symmetry of these relations, $(t,s) \in R_1$ and $(t,s) \in R_2$. The definition of intersection now gives $(t,s) \in R$, as needed.

Transitivity. Let $(s,t),(t,u) \in R$. Then (s,t) and (t,u) both lie in R_1, and by transitivity of this relation $(s,u) \in R_1$ as well. Similarly, $(s,u) \in R_2$. Thus by definition of intersection, $(s,u) \in R_1 \cap R_2 = R$, and we're done.

(b) The simplest example of two equivalence relations whose union is not an equivalence relation involves two relations on a set with 3 elements. Let $S = \{a,b,c\}$ and

$$R_1 = \{(a,a),(b,b),(c,c),(a,b),(b,a)\} \quad \text{and} \quad R_2 = \{(a,a),(b,b),(c,c),(b,c),(c,b)\}.$$

Then R_1 and R_2 are both equivalence relations (this is not hard to check), but their union,

$$R_1 \cup R_2 = \{(a,a),(b,b),(c,c),(a,b),(b,a),(a,c),(c,a)\},$$

is not transitive: it should contain (a,c), for instance, since (a,b) and (b,c) both lie in it.