

Exam 2

Working by yourself, please answer each question below on your own paper. You may use your class notes (but please no other resources!), and you may consult with me (but I promise not to be too helpful!). You may get help neither from each other nor from Math Lab staff nor from any other faculty or staff at UNC Asheville.

The exam is worth a total of 50 points, and the point value of each question is given with that question. Your completed first draft of the exam is due by 5:00 p.m. on **Thursday, April 2nd**. Upon receiving your roughly graded exam you will have an opportunity to perform revisions and submit a final version on **Friday, April 10th**.

- (1) (15 points) This exercise will justify the term “binomial coefficients” for the numbers $\binom{n}{k}$. Prove that for any terms a and b and for any $n \in \mathbb{N} \cup \{0\}$,

$$(a + b)^n = \binom{n}{0}b^n + \binom{n}{1}ab^{n-1} + \binom{n}{2}a^2b^{n-2} + \cdots + \binom{n}{n-1}a^{n-1}b + \binom{n}{n}a^n = \sum_{k=0}^n \binom{n}{k}a^k b^{n-k}.$$

That is, the terms $\binom{n}{k}$ appear as coefficients in the expansion of integer powers of a binomial.

(*Hints*: induct on n , and use the fact that $(a + b)^{n+1} = (a + b)^n \cdot (a + b)$. If you’re not a fan of sigma sums, you may prefer to work with the “long-hand” version of the sum above.)

- (2) (5 points) Prove that if A is a set, then the relation $R = A \times A = \{(a, b) \mid a, b \in A\}$ is always an equivalence relation on A . Describe in your own words what it means for two elements a and b to be equivalent modulo this relation. (This R is one of two “trivial” equivalence relations on the set A .)
- (3) (10 points) Prove carefully why it is that a full house beats a flush and a flush beats a an ordinary straight (*i.e.* not a straight flush) in poker. (For simplicity you may assume that aces cannot be used as low cards, only as high cards, though it turns out that this doesn’t affect the result.) Explain all of your computations carefully!
- (4) (10 points) Decide whether the following statement is true or false; if it true, prove it, and if it is false, provide a counterexample to demonstrate that fact:

For any sets A , B , and C , $(A \setminus B) \times C = (A \times C) \setminus (B \times C)$.

(*Hint*: approach this problem very carefully, breaking it down as needed using the definitions of “ \setminus ” and “ \times .”)

- (5) (10 points) Suppose that we live in a city built on a grid system of streets: all roads run either north-south or east-west. Suppose that your favorite restaurant, Salami Slim’s Olde Vegan Sandwiche Shoppe, lies 7 blocks west of your apartment, and 5 blocks north. Assuming you can only walk along streets, how many different *shortest* paths can you take from your place to the restaurant? Please explain any computations carefully, justifying them with the proper combinatorial techniques. (*Hint*: is there a way of “recording” a given path so that you can count the number of paths you may construct?)