

Exam 1

Working by yourself, please answer each question below on your own paper. You may use your class notes (but please no other resources!), and you may consult with me (but I promise not to be too helpful!). You may get help neither from each other nor from Math Lab staff nor from any other faculty or staff at UNC Asheville.

Keep in mind that neatness is nice, and that complete sentences are crucial to well-composed writing!

The exam is worth a total of 50 points, and the point value of each question is given with that question. Your completed first draft of the exam is due by 5:00 p.m. on **Friday, February 27th**. Upon receiving your roughly graded exam you will have an opportunity to perform revisions and submit a final version on **Friday, March 6th**.

- (1) (18 points total) We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *bounded* if there are real numbers m and M such that for all $x \in \mathbb{R}$, $m \leq f(x) \leq M$.
 - (a) (4 points) Write the definition of a bounded function, given above, in quantifier notation. You should begin with “ $f : \mathbb{R} \rightarrow \mathbb{R}$ is *bounded* if...”
 - (b) (7 points) **True or false?** Every continuous function is bounded. (Please prove your assertion.)
 - (c) (7 points) Prove that the sum of two bounded functions is a bounded function.
- (2) (10 points) Let’s define the numbers a_1, a_2, \dots by letting $a_1 = a_2 = 1$ and

$$a_n = \frac{1}{2} \left(a_{n-1} + \frac{2}{a_{n-2}} \right)$$

for all $n \geq 3$. Prove that for all $n \in \mathbb{N}$, $1 \leq a_n \leq 2$. (*Hint*: prove $1 \leq a_n$ and $a_n \leq 2$ separately.)

- (3) (12 points total; 6 points each) Recall that the L_1 -tile is a small “L” shaped tile with sides as shown below:

- (a) Describe the set $S \subseteq \mathbb{N}$ such that $n \in S \Leftrightarrow$ the rectangular “checkerboard” of size $2 \times n$ can be covered by non-overlapping L_1 -tiles. Please prove your assertion.
- (b) Describe the set $T \subseteq \mathbb{N}$ such that $n \in T \Leftrightarrow$ the rectangular “checkerboard” of size $3 \times n$ can be covered by non-overlapping L_1 -tiles. Please prove your assertion.

- (4) (10 points total; 5 points each) The sets \mathbb{Z} , \mathbb{Q} , and \mathbb{R} are all examples of the sort of mathematical object called a *ring*. One fundamental property of a ring R is that it's got two operations, often denoted $+$ and \cdot and called *addition* and *multiplication*, under which it's *closed*. That is, the *sum* of any two objects in the ring is in the ring, and the *product* of any two objects in the ring is in the ring.

Consider the set R defined by

$$R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}.$$

That is, R is the set of sums where one term is an integer and the other is an integer multiple of $\sqrt{2}$.

- (a) Prove that R is closed under sums.
- (b) Prove that R is closed under products.