Compounding Statements

There are other ways, besides quantification and negation ("not"), to create new statements from old ones. Four of the next most common *connectives* are

(1) *conjunction*: "P and Q," written $P \wedge Q$,
(2) *disjunction*: "P or Q," written $P \vee Q$,
(3) *biconditional*: "P if and only if Q" (sometimes lazily abbreviated "P iff Q"), written $P \Leftrightarrow Q$, and
(4) *conditional*: "P implies Q," written $P \Rightarrow Q$.

More terminology: in the conditional statement $P \Rightarrow Q$, P is known as the *hypothesis* and Q, the *conclusion*. This terminology makes sense, because in this case we can **conclude** Q by **hypothesizing** P.

By the way, the statement $Q \Rightarrow P$ is known as the *converse* of $P \Rightarrow Q$.

An elementary way to keep track of the truth values of compound statements is by means of a *truth table*. Using your intuition (often a good mathematical guide!), you ought to be able to fill in the values missing from the following such table:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Leftrightarrow Q$	$P \Rightarrow Q$
T	T	F	T			
T	F		F			F
F	T					
F	F				T	

The trickiest entries in the table fall under the conditional column. Note that this is the only column whose values change if we exchange the positions of P and Q in the statement we're making.

We can build such a truth table to determine the truth value of compound statements like $(P \Rightarrow Q) \Leftrightarrow [(\neg P) \vee Q]$. Letting R represent this statement, fill in the necessary values below to find the truth values of R For all possible values of P and Q:

P	Q	$P \Rightarrow Q$	$\neg P$	$(\neg P) \vee Q$	R
T	T				
T	F				
F	T				
F	F				

Given so many different compound statements we could make, it should be no surprise that there are some *logically equivalent* statements (*i.e.*, statements with the same truth values) that look different at first. A few of the most important equivalent pairs are listed below:

(1) $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$,
(2) $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$,
(3) $\neg(P \Rightarrow Q)$ and $P \wedge (\neg Q)$,
(4) $P \Leftrightarrow Q$ and $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$,
(5) $P \vee Q$ and $(\neg P) \Rightarrow Q$, and finally
(6) $P \Rightarrow Q$ and $(\neg Q) \Rightarrow (\neg P)$.

In the space below, put together a truth table that shows both 1 and 5:

Why care about these equivalences? They will often help us prove things! Notice, for instance, what 4 is saying: in order to prove that P and Q have the same truth value (that is, that one is true if and only if the other is), we really end up showing that P implies Q and *vice versa*.

Similarly, 6 tells us that in order to prove that P implies Q, we could manage the same feat by showing the *contrapositive* statement $(\neg Q) \Rightarrow (\neg P)$.

By the way, the first two equivalences above are important enough to have their own names. Collectively they are known as the *De Morgan Laws*, after the 19th century British logician and mathematician Augustus De Morgan.