You Can Count on It, II:
Electric Booga...er...Combinations!

We ended the last handout with the following tricky problem:

Qmail passwords are made by combining English alphabet letters and decimal digits, and are not case-sensitive, so that "ZOmg" and "zoMg" would count as the same thing. How many Qmail passwords are possible if each password must contain at least 4 characters and at most 6, and each password must contain at least 1 decimal digit, but at most 2? You *are* allowed to repeat characters within the password. (*Hint*: you'll need both Addition and Multiplication Principles here...)

Computing the number of valid passwords involved using the Addition Principle to break up the set of passwords into passwords of length 4, of length 5, and of length 6. From there, we had to figure out a way of counting the number of different "patterns" which letters and digits could make, a step which required the Addition Principle again, before finally applying the Multiplication Principle at the final stage.

For instance, in the case of length-6 passwords, we had the following valid patterns of letters and digits:

$$DLLLLL,\quad LDLLLL,\quad LLDLLL,$$
$$LLLDLL,\quad LLLLDL,\quad \text{and } LLLLLD$$

when there's 1 digit, and

$$DDLLLL,\quad DLDLLL,\quad DLLDLL,\quad DLLLDL,\quad DLLLLD,$$
$$LDDLLL,\quad LDLDLL,\quad LDLLDL,\quad LDLLLD,\quad LLDDLL,$$
$$LLDLDL,\quad LLDLLD,\quad LLLDDL,\quad LLLDLD,\quad \text{and } LLLLDD$$

when there are 2 digits!

Now *come on*, there's *gotta* be an easier way to count these possibilities!

Good news! There is an easier way. We'll develop our approach by considering a more general problem which turns out to apply in our case:

Problem. How many different ways are there of creating a set with k distinct elements in it by selecting those elements from a set with n elements? Put another way, how many *distinct* (*not* disjoint!) subsets of size k are there inside of a set of size n? A subset with k elements is called a *k-combination* of the set S.

Proposition. Let $|S| = n$ and let $0 \le k \le n$. The number of subsets of S of size k is given by $\frac{n!}{(n-k)!k!}$, where $n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n$ is the *factorial* function. (Strangely enough, $0! = 1$. Go figure.)

Example. Verify that this formula is true for all k, $0 \leq k \leq 3$, if $|S| = 3$. (You can do this just by finding all possible subsets!)

Example. Use the formula to find the number of subsets of size 3 in a set of size 8, and the number of subsets of size 4 in a set of size 8.

The numbers $\frac{n!}{(n-k)!k!}$ show up often enough in mathematics to have special notation and terminology. They're called *binomial coefficients* (you'll see why in your next homework assignment), and are denoted as follows:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

This is generally read "n choose k," since we're giving the number of ways of *choosing* k elements from among n.

Before we verify that our formula is correct (we *still* haven't done that!), let's see how to use it to solve our password problem.

Notice that any "pattern" of letters and digits of length 6 can be made to naturally correspond to a subset of $S = \{1, 2, 3, 4, 5, 6\}$ in the following way: for any pattern of Ls and Ds, we choose the subset of S containing the positions where we see Ds. For instance,

$$LLDLDL \quad \text{corresponds with} \quad \{3, 5\} \subseteq S.$$

Therefore, figuring out how many patterns there are with 1 digit amounts to figuring out how many subsets of S there are with 1 element. There are $\binom{6}{1} = 6$ such subsets, so there are 6 patterns with a single digit.

Exercise. Compute the number of patterns with

(1) 6 characters and 2 digits, and 6 characters and 1 digits:

(2) How about if we allowed ourselves 7 characters and 1 digit? 2 digits? 3 digits?

(3) How about with 8 characters and 1 digit? 2 digits? 3 digits?

This is a pretty powerful counting technique! Let's check that the formula actually works:

Proof. We prove this by induction on n.

Base case. Say $|S| = 0$; that is, $S = \emptyset$. How many subsets of size $k = 0$ are there?

Inductive hypothesis. Let's suppose that we've shown that if $|S| = n$ and $0 \leq k \leq n$, S has $\binom{n}{k}$ subsets of size k.

Inductive step. Let's let $|S| = n + 1$ and take any k between 0 and $n + 1$. We'll simply *count* the number of subsets of size k by taking two cases and using the Addition Principle.

If $S = \{s_1, s_2, ..., s_n, s_{n+1}\}$, we can consider two kinds of sets of size k: those that *don't* contain s_{n+1} and those that *do*.

How many of the first sort of set are there? (*Hints*: how big are such sets? Is there a set in which they lie to which we could apply the inductive hypothesis?)

How many of the second sort of set are there? (*Hints*: what happens if you *remove* s_{n+1} from such a set? How big is the remaining set? Where does it lie?)

You should have found $\binom{n}{k}$ of the first sort, and $\binom{n}{k-1}$ of the second, by inductive hypothesis applied to $S \setminus \{s_{n+1}\}$. We can now add these values to obtain the number of subsets of S of size k (be careful with those common denominators!):

The following fact follows from the way in which we proved our formula:

Theorem. For any $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

Proof. The sum counts all subsets of a set S, $|S| = n$, by breaking them down by size. We already showed that the powerset of S has size 2^n. Thus the two must be equal! ♢

Let's finish with a couple more examples to demonstrate the usefulness of this formula:

Examples.

(1) There are 26 people enrolled in our class, and on average 21 people attend class each day. How many *different* sets of people might I find greeting me on a given day?

(2) How many different 5-card poker hands can be dealt from a 52-card deck?

(3) How many 5-card poker hands contain a pair, but *nothing better* than this? (*Hint*: you'll want to compute the number of pairs that can appear, and then figure out the number of ways of filling in the remaining 3 cards...)

All well and good! We'll next finish off our brief introduction to combinatorics by considering the related issue of *permutations*.