You Can Count on It:

Basic Combinatorial Techniques

Combinatorics is the mathematical art of counting. Since counting is one of the first mathematical tasks most of us mastered, it might seem weird that we can learn new ideas concerning counting this late in the game. Nevertheless, combinatorics is a hotly researched field of mathematics, as mathematicians spend boundless energy in attempting to answer the question *How many...?*

The following are questions concerning combinatorial reasoning:

(1) Why does a full house beat a straight, while a straight-flush beats a full house?
(2) A committee of five people, to be led by a chair with special duties, is to be formed from 15 people. How many different ways are there of making up such a committee?
(3) Our house lies five blocks south and seven blocks east of our favorite restaurant. Our hometown's roads (unlike those of Asheville) are laid out along a perfect grid system, and there are no one-way streets. How many *different shortest* ways are there of walking from our house to the restaurant?

To begin our approach problems like these, we focus on three simple tools for counting:

(1) *The Pigeonhole Principle*,
(2) *The Addition Principle*, and
(3) *The Multiplication Principle*.

From there we'll consider the basics of permutations and combinations, techniques that will give us much more mastery of sets and the objects they possess as elements.

Round 1: The Pigeonhole Principle.

The name of this almost axiomatic result of comes from antique desks possessing *pigeonholes* into which small objects can be placed for easy access. The principle is easy to state, and not very difficult to use:

The Pigeonhole Principle. Suppose at least $n+1$ objects are to be placed in n boxes. (That is, at least $n+1$ elements are to be placed in n sets.) Then there is some box containing at least two elements.

Proof. Suppose by way of contradiction every one of our n sets has at most 1 element in it. Then the total number of objects can be no more than $n \cdot 1 = n$, a contradiction. $\diamondsuit$

Here's a little room for you to provide a "picture" of the Pigeonhole Principle:

Examples. Let's use the Pigeonhole Principle right away. In each of the problems below, be sure to indicate what your "objects" are, and how your "boxes" are labeled.

(1) How many people must be in a class so that at least two people in the class were born in the same month?

(2) Show that any five-card poker hand will have at least two cards of the same suit.

These examples are quite simple, but we can spice them up a little bit by using the

Strong Form of the Pigeonhole Principle. If at least $kn + 1$ objects are to be placed in n boxes, then there is some box containing at least $k + 1$ elements.

Notice that the original form of the principle is simply the strong form, with $k = 1$. The proof of the strong form is similar enough to the first proof that you should be able to handle it yourself:

Proof. Suppose...

Example. Suppose there are 101 people at a large party at The Grove, and that each person at the party has an *even* number (possibly 0) of acquaintances at the party, excluding her/himself. Prove that there are at least 3 people with the same number of acquaintances. (Notice you can't guarantee what that common number will be!)

In the homework you'll be asked to use a little cleverness to show that you can perform the same feat with only 100 people.

The Addition Principle.

To begin with, we need one more

Definition. A *partition* of a set S is a collection $S_1, S_2, ..., S_n \in \mathcal{P}(S)$ of *disjoint* subsets of S such that

$$S = S_1 \cup S_2 \cup \cdots \cup S_n.$$

(Notice now that we know $\cup$ is *associative*, it doesn't matter in what order we perform our unions!) These subsets S_i are often called *parts* or *classes* in the partition.

Example. How many different partitions are there of the set $\{1, \pi, e\}$?

The Addition Principle. Suppose $S_1, S_2, ..., S_n$ is a partition of the set S. Then

$$|S| = |S_1| + |S_2| + \cdots + |S_n| = \sum_{i=1}^{n} |S_i|.$$

This simple principle often comes in handy in breaking a large counting problem into smaller ones, and is particularly useful when combined with The Multiplication Principle, which we'll talk about in a moment.

Example. Files containing U.S. Census data from 2000 have been corrupted, and though the population figures for North Carolina have been wiped out, the figures for every county individually are intact. How can the lost data be recovered?

The Multiplication Principle.

Let's start off with an

Example. Suppose North Carolina's standard-issue license plate numbers consist of a string of three letters (not necessarily distinct), followed by a string of four decimal digits (not necessarily distinct) from the set $\{0, 1, 2, ..., 9\}$. How many different license plate numbers can be generated?

In solving this problem, you almost certainly made use of the

The Multiplication Principle. If S and T are any finite sets, then

$$|S \times T| = |S||T|.$$

Recall that here $S \times T$ denotes the Cartesian product of S and T, the set of all ordered pairs of the form (s, t), $s \in S$ and $t \in T$.

Here are some more

Examples.

(1) I have 16 pairs of socks and 3 pairs of shoes. How many different ways can my feet be dressed up, assuming that my socks and shoes must match as pairs?

(2) Come up with a maximum for the number of two-letter words possible using the English alphabet, if every word must consist of at least one vowel. (Let's let "y" always count as a consonant. *Hint*: you'll probably want to use The Addition Principle, too...)

This principle only gets really fun when we allow ourselves to form Cartesian products *of* Cartesian products. Note that we can use the definition of $\times$ to compute the following set

$$(S \times T) \times V = \Big\{ \big((s,t),v\big) \mid s \in S, t \in T, \text{ and } v \in V \Big\}$$

of ordered 3-*tuples*. More generally, we could form sets of ordered n-*tuples* $(s_1, s_2, ..., s_n)$ by taking the Cartesian product of n sets.

Once we know a little bit more about functions, we'll be able to show that $(S \times T) \times V$ and $S \times (T \times V)$ are essentially the same set, so it doesn't matter in what order we do our multiplying. Let's take this on faith for the moment, and generalize The Multiplication Principle:

Generalization. The Cartesian product of the sets $S_1, S_2, ..., S_n$ taken in *any* order has cardinality equal to

$$|S_1| \cdot |S_2| \cdot \cdots \cdot |S_n|.$$

Examples.

(1) At Salami Slim's Ye Olde Vegan Sandwiche Shoppe, customers can build a half-meter-long sandwich by selecting one of 6 different breads, one of 4 different meat substitutes, one of 13 different vegetable toppings, and one of 5 different condiments. How many different ways can such a sandwich be made?

(2) Qmail passwords are made by combining English alphabet letters and decimal digits, and are not case-sensitive, so that "ZOmg" and "zoMg" would count as the same thing. How many Qmail passwords are possible if each password must contain at least 4 characters and at most 6, and each password must contain at least 1 decimal digit, but at most 2? You *are* allowed to repeat characters within the password. (*Hint*: you'll need both Addition and Multiplication Principles here, and maybe a little extra ingenuity...)

We'll continue our counting adventure by considering permutations and combinations next!