MATH 280, Section 1
Foundations of Mathematics
Fall 2009

The Basics of Sets

Now that we've got some basic techniques for proving mathematical statements, we need to develop some mathematical objects about which we can *make* statements to prove. The most basic of all mathematical objects is a *set*.

Definition. For right now, a *set* is nothing more than an unordered collection of objects, called its *elements*.

Sets can be denoted with any sort of notation, but frequently we'll use uppercase Roman letters: A, B, C, *etc.* (See Polya's entry on *Notation* in the text for class!) Elements of a set are generally written with the corresponding lowercase letters: $A = \{a_1, a_2, , a_3, ...\}$, or $A = \{a, a', a'', ...\}$.

As we can see from the last paragraph often the elements of a set are indicated simply by listing them. Even if a set is *infinite* (that is, has infinitely many elements; we'll make this more precise later), we can often "list" it by establishing a pattern with the first few elements.

Examples. You should list the following sets below:

- The set of prime numbers less than 20:
- A set of trig functions with period 2π:
- The set of even natural numbers:
- The set of American state names beginning with the letter 'A':

As we know already, we indicate that a is an element of A with the notation $a \in A$. Thus, $2 \in \mathbb{Z}$ and California $\in$ { California, Connecticut }.

Definitions. The set A is said to be a *subset* of the set B, or to be *contained in* B (written $A \subseteq B$) if $a \in A \Rightarrow a \in B$. How could you write this in quantifiers?

If both $A \subseteq B$ and $B \subseteq A$ hold, we say that A and B are *equal*, and we write $A = B$. So to show that two sets are equal is to show that one contains the other, and *vice versa*.

Example. Suppose the set S of real numbers is defined by the following rules:

1. First we place $1 \in S$.
2. At each successive step, we place $s + 1$ in S if and only if $s \in S$ already.

What set is S at the end of time? Can you prove it?

If we are given a set S, we often indicate a particular subset of S by specifying a property we want the elements of the subset to satisfy. If P is the given property, the set

$$\{s \in S \mid P(s)\}$$

is the set of $s \in S$ satisfying P.

Examples. Find the following sets:

- $\{$ *state* $\in$ USA $\mid$ *state* has four letters in its name $\}$:

- $\{x \in \mathbb{N} \mid x \leq 100 \text{ and } x \text{ is divisible only by } \pm x, \pm 1, \text{ and } \pm 2\}$:

Definition. There is a unique set, called the *empty set* or *null set* and denoted $\emptyset$ or $\{\ \}$, that has no elements in it.

Question. Let S be any set. Is $\{\ \} \subseteq S$ true? Prove your claim.

Definition. If $A \subseteq B$ and there is some element $b \in B$ such that $b \notin A$, we say that A is a *proper subset* of B, and denote this by $A \subset B$.

Examples. Let $S = \{1, 3, 5\}$.

- List all proper subsets of S. (Make sure you don't overlook one particularly simple subset!)

- List *all* subsets of S.

- Prove that if $A = \{x \in \mathbb{R} \mid x^2 - 4x + 3 = 0\}$ and $B = \{x \in \mathbb{N} \mid 1 \leq x \leq 4\}$, then $A \subseteq B$. Can you say more?

Definition. The collection of all subsets of a given set S is important enough to give it a name in its own right. We call this set the *powerset* of S, and it's often denoted $\mathcal{P}(S)$.

Examples. Find the powersets for each of the following sets:

- $\{\ \}$

- $\{1\}$

- $\{1, 2\}$

- $\mathcal{P}(\{1, 2\})$

- $\{1, 2, 3\}$

- $\{1, 2, 3, 4\}$

Do you have a conjecture as to the size of $\mathcal{P}(S)$? Can you prove it?

We finish with a couple of basic *binary operations* on sets (we'll cover a few more in the next handout):

Definitions. Given two sets A and B, the *union* of A and B, denoted $A \cup B$, is the set of all elements in *at least one of* A and B:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

the *intersection* of A and B, denoted $A \cap B$, is the set of all elements in *both* A and B:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Examples. Compute the sets resulting from the indicated operations:

1. $\begin{array}{c}\{\ \textit{state}\ \in\ \text{USA} \mid \textit{state}\text{ has four letters in its name }\}\cup \\ \{\ \textit{state}\ \in\ \text{USA} \mid \text{name of }\textit{state}\text{ begins with O }\}\end{array}$:

2. $\{x \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})\ x = 2k\} \cap \{x \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})\ x = 2k + 1\}$:

3. $\{x \in \mathbb{R} \mid e \leq x \leq \pi\} \cap \mathbb{N}$:

4. $\{x \in \mathbb{N} \mid x^2 - 1 > 0\} \cup \{-1, 0, 1\}$: