More Induction Examples!

I thought I'd make up this handout in order to provide a couple more examples of inductive proofs, including another involving sigma sum notation, since I know that's a sticky point for folks who don't deal with sigma sums all that often.

Please note that if sigma notation is not familiar to you, or if you have trouble remembering how it works, *let me know immediately*, and we can go over a few more examples, do a little extra, reading, *et cetera*. You will definitely *need to know* how to work with sigma sums well.

Proposition. If $n \in \mathbb{N}$, then

$$\sum_{k=1}^{n} 2^{k-1} = 2^n - 1.$$

Note: Here k is the *index* of the sum, while n is the variable over which we will induct.

Proof. We prove the proposition by induction on n.

Base case. Let $n = 1$. Then

$$\sum_{k=1}^{n} 2^{k-1} = \sum_{k=1}^{1} 2^{k-1} = 2^{1-1} = 2^0 = 1.$$

Meanwhile,

$$2^n - 1 = 2^1 - 1 = 2 - 1 = 1.$$

Having shown the result holds for $n = 1$, the base case is established.

Inductive hypothesis. Assume that the result holds for a given value of n.

Inductive step. Let us now consider $n + 1$. Beginning with the left-hand side, we pull off the last term and write it separately:

$$\sum_{k=1}^{n+1} 2^{k-1} = \sum_{k=1}^{n} 2^{k-1} + 2^{(n+1)-1}.$$

By inductive hypothesis, the remaining sigma sum is equal to $2^n - 1$. Thus we obtain

$$2^n - 1 + 2^{(n+1)-1} = 2^n + 2^n - 1 = 2^{n+1} - 1,$$

which is what we needed to show. ◇

Note: At this point, I would very much like you to *clearly* indicate each step in your inductive proofs, as is done above!

To understand our next example, we need to recall that the *Fibonacci sequence* is the list of numbers

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...\},$$

wherein after the first two terms, each subsequent term in the list is obtained by adding the previous two together. We denote the n term in the list by F_n. For example, $F_1 = 1$, $F_2 = 1$, $F_3 = 1+1 = 2$, $F_4 = 1 + 2 = 3$, and so forth.

Proposition. F_n is even if and only if n is divisible by 3.

Proof. We induct on the number n.

Base case(s). Clearly $F_1 = 1$ and $F_2 = 1$ are odd, and $F_3 = 2$ is even. Thus for $n = 1, 2, 3$, F_n is even if and only if 3 is divisible by 3, since 3 is the only one of these numbers with that property.

Inductive hypothesis. By *Strong* Induction, suppose that for all $k \leq n$, F_k is even if and only if n is divisible by 3.

Inductive Step. Now consider $n+1$; we must show it is divisible by 3 precisely when F_{n+1} is even. We have three cases to consider:

Case 1: $n + 1$ is divisible by 3. In this case, neither $n - 1$ nor n is divisible by 3. (Can you prove this carefully? Perhaps a proof by contradiction is in order!) By the inductive hypothesis, therefore, both F_{n-1} and F_n are *odd*. (Note we can apply the hypothesis to both of these numbers, since $n - 1 \leq n$ and $n \leq n$ both hold.) Thus $F_{n+1} = F_n + F_{n-1}$ is the sum of two odd numbers, and is therefore even.

Case 2: $n+1$ is not divisible by 3, but n is. In this case, $n-1$ is not divisible by 3 (again, proof?), so by inductive hypothesis, F_n is even, but F_{n-1} is odd. Thus $F_{n+1} = F_n + F_{n-1}$ is the sum of an even number and an odd number, and is therefore odd.

Case 3: neither $n + 1$ nor n is divisible by 3, so that $n - 1$ must be (proof?). In this case, the inductive hypothesis says that F_n is odd and F_{n-1} is even, so that again $F_{n+1} = F_n + F_{n-1}$ is the sum of an odd number and an even number, and is therefore odd.

$\diamondsuit$