Counting k-combinations

Theorem. Let $n \geq 0$ and let $|S| = n$. Then the number of k-combinations in S is $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, for any k such that $0 \leq k \leq n$.

Proof. We will prove this by induction on n.

Base Case. If $n = 0$, then $S = \emptyset$. The only value of k which me must consider is $k = 0$. Notice that there is 1 subset of S of size 0 (namely $\emptyset$ itself). Moreover,

$$\binom{0}{0} = \frac{0!}{0!0!} = \frac{1}{1 \cdot 1} = 1,$$

so the base case is established.

Inductive hypothesis. Assume for a fixed and arbitrary n that if $|S| = n$ then there are $\binom{n}{k}$ subsets of size k in S.

Inductive Step. Now suppose that $|S| = n + 1$, and let $S = \{s_1, s_2, ..., s_n, s_{n+1}\}$. We will count the subsets T of S by breaking them into two different cases: (1) those T such that $s_{n+1} \notin T$ and (2) those T such that $s_{n+1} \in T$.

Case 1. Suppose $s_{n+1} \notin T$. Then not only is T a subset of S, it's *also* a subset of $S' = S \setminus \{s_{n+1}\}$. Since $|S| = n + 1$, $|S'| = n + 1 - 1 = n$. Thus, by inductive hypothesis, T is actually one of the $\binom{n}{k}$ subsets of S' of size k!

Case 2. Suppose $s_{n+1} \in T$. Note that T is completely determined by the set $T' = T \setminus \{s_{n+1}\}$. That is, if we know what T' is, we can obtain T from it just by re-including s_{n+1}, and we can count the subsets T of size k by instead counting the subsets $T' = T \setminus \{s_{n+1}\}$ of size $k - 1$ without s_{n+1}. This is illustrated below, where $n + 1 = 3$: on the left are the sets T' without $s_{n+1} = s_3$, and on the right are the corresponding sets T with $s_{n+1} = s_3$ re-included:

T'	T
$\emptyset$	$\{s_3\}$
$\{s_1\}$	$\{s_1, s_3\}$
$\{s_2\}$	$\{s_2, s_3\}$
$\{s_1, s_2\}$	$\{s_1, s_2, s_3\}$

By inductive hypothesis, since any such T' (where $|T'| = k - 1$ and $s_{n+1} \notin T'$)) is a subset of $S' = S \setminus \{s_{n+1}\}$, we know that there are $\binom{n}{k-1}$ such sets T'. Thus there are exactly as many subsets T of size k which have s_{n+1} as an element.

Now we applying the Addition Principle: the total number of subsets of size k is obtained by adding the numbers of such sets found in the cases above. Thus the number we seek is

$$\begin{aligned}
\tbinom{n}{k} + \tbinom{n}{k-1} &= \tfrac{n!}{(n-k)!k!} + \tfrac{n!}{(n-(k-1))!(k-1)!} \\
&= \tfrac{n!(n-k+1)}{(n-k)!k!(n-k+1)} + \tfrac{n!k}{(n-k+1)!(k-1)!k} \\
&= \tfrac{n!(n-k+1)}{(n-k+1)!k!} + \tfrac{n!k}{(n-k+1)!k!} \\
&= \tfrac{n!(n-k+1+k)}{(n-k+1)!k!} \\
&= \tfrac{(n+1)!}{((n+1)-k)!k!} \\
&= \tbinom{n+1}{k},
\end{aligned}$$

which is what we wanted.