MATH 280, Section 1
Foundations of Mathematics
Fall 2009

The Principle of Mathematical Induction
(a.k.a. Domino Theory)

Suppose we wish to verify a formula like Gauss's identity

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

for *any* positive integer n. There are a number of ways of doing this, including geometric and arithmetic tricks that work for any given number. Here, the proof method we will use is called induction.

Induction is one of the most important techniques of mathematical proof. It is used to verify claims made about infinitely many related statements like the statements, one for each integer n, made above. The basic idea is a simple one, and consists of three easy steps:

1. Prove that the statement you wish to make holds in some initial *base case*,
2. prove that if the statement holds for the nth case, then it also holds for the $(n+1)$st case, and from these facts
3. conclude that the statement holds for the 1st (base) case, the 2nd case, the 3rd case, and so on.

Induction is often compared to dominos, for the obvious reason: the truth of any one of our statements (except for the base case) follows from the truth of the previous statement, as in Step 2 above. Let's do an

Example. Prove that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for any positive integer n.

Proof. Direct computation shows that the statement is true for $n = 1$. It's also easy to verify for $n = 2$, $n = 3$, and so forth. Let's simply *assume* the statement is true for a fixed, *arbitrary* value of n, and let's verify that it's then true for $n + 1$, the next value.

Well, $\displaystyle\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1)$. But by our assumption that the statement is true for n, this last is equal to $\frac{n(n+1)}{2} + (n+1)$. Finding a common denominator, we complete the calculations:

$$\frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+2)(n+1)}{2}.$$

Simply permuting the terms in the numerator, we see we have $\frac{(n+1)[(n+1)+1]}{2}$, the formula we wanted, corresponding to $n + 1$. ♢

Why are we done at this point? Suppose the number we were really interested in was 178, and that $P(178)$ is the statement we need to verify. We've shown that if $P(177)$ is true, then so is $P(178)$. But we also know that if $P(176)$ is true, then so is $P(177)$, and thus $P(178)$. But we can step back again, reducing to $P(175)$, and then to $P(174)$, and so on...all the way back to the base case, $P(1)$, which we know is true because we checked it!

Before we go any further, let's outline the

Anatomy of an Inductive Proof

A proof by mathematical induction has three essential elements:

1. The proof of a *base case*, much like $P(1)$ above,
2. the assumption of an *inductive hypothesis*: "Assume that we have proven $P(n)$...", and
3. the proof of the *inductive step*, showing that *if* $P(n)$ is true, then so is $P(n+1)$.

That's all there is to it! Now it's your turn. Try your hand at your first inductive proof (it might help to simply follow the steps laid out above in the "anatomy"):

Proposition. For any $n \in \mathbb{N}$, $1 + 3 + 5 + \cdots + (2n+1) = n^2 + 2n + 1$. (That is, the sum of the first $n+1$ odd natural numbers is given by $n^2 + 2n + 1$.)

Note that $n^2+2n+1 = (n+1)^2$; thus, the sum of the first $n+1$ natural numbers is a perfect square! This fact, derived easily from our previous, more complicated proof, is known as a *corollary.*

Try proving the following

Proposition. If $n \in \mathbb{N}$ and $a \in \mathbb{R}$ such that $a \geq 2$, then $n < a^n$. (This is a statement about the growth of the exponential function $f(x) = a^x$, without calculus!)

(*Hint*: in the inductive step, when $n \geq 2$ we get $n + 1 < n + n = \cdots$)

See how easy?

Technically, there are two kinds of induction. So far we've been focussing on the form known as *Weak Induction*, in which our inductive hypothesis is "$P(n)$ is true." The form known as *Strong Induction* allows us to assume something more; here our inductive hypothesis is "$P(k)$ is true *for all* $k \leq n$."

Clearly Strong Induction implies Weak Induction, since we assume $P(n)$ in both cases. Here's some room for us to verify that in fact "Weak" Induction is really just as strong as Strong Induction. Let's do it! (*Hint*: given the statement "P," let $Q(n)$ be the statement "$P(k)$ is true for all $k \leq n$.")

Here's an example where Strong Induction comes in handy:

Proposition. Every nonempty subset of $\mathbb{N}$ has a least element. (We thus say that $\mathbb{N}$ is *well ordered.*)

Proof. Let $P(n)$ be the statement "if $S \subseteq \mathbb{N}$ contains n, then S has a least element. We prove $P(n)$ is true for all $n \in \mathbb{N}$.

$\diamondsuit$

We should also note that with induction we can prove some statements that are not quite so "numerical" in nature. For example, our "play" in class on Wednesday shows the truth of the following fact:

Proposition. Let L_n denote the L-shaped "tile" with side lengths shown in the figure below. Then for any $n \geq 1$, the tile L_n can be constructed by pasting together non-overlapping copies of L_1.

Here are a couple of hints to keep in mind when dealing with induction proofs:

1. Be *very* careful in proving your base case, to make sure that it really *is* the correct base case. One of the exercises from your homework will illustrate this pitfall well.
2. Note that our base case need not always correspond to $n = 1$; it could be that we are asked to show a mathematical fact for all numbers above some $n \neq 1$. Your homework will contain too an example where this is the case.