

Greatest Hits, Volume III: Homework 4.2 and 4.4

2. Prove that $2^n \leq n!$ for every $n \in \mathbb{N}$ satisfying $n \geq 4$.

Proof: Assume that two consecutive perfect cubes are represented the numbers n and m . We call a number n a *perfect cube* if it is the cube of an integer: $n = k^3$, for some $k \in \mathbb{Z}$. The difference between two consecutive perfect cubes is therefore $m - n$.

$(k+1)^3 - k^3 = k^3 + 3k^2 + 3k + 1 - k^3 = 3k^2 + 3k + 1 = 2k^2 + k^2 + 2k + k + 1 = [2(k^2 + k) + 1] + k^2 + k$.
 $[2(k^2 + k) + 1]$ is odd by the definition of an odd number, so we must prove that $k^2 + k$ is even in order to prove that the difference between two consecutive perfect cubes is odd, since the sum of an odd number and an even number is odd.

In the case that $k \in \mathbb{Z}$, and is odd; let $k = 2x + 1$.

$$(2x + 1)^2 + (2x + 1) = 4x^2 + 6x + 2 = 2(2x^2 + 3x + 1).$$

$2(2x^2 + 3x + 1)$ is even by definition of an even number.

In the case that $k \in \mathbb{Z}$, and is even; let $k = 2x$.

$$(2x)^2 + (2x) = 2(2x^2 + x).$$

$2(2x^2 + x)$ is even by the definition of an even number, meaning that $[2(k^2 + k) + 1] + k^2 + k$, which represents the difference between two consecutive perfect cubes, is odd.

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4. Prove that for any $n \in \mathbb{N}$,

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

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First, let us look at the case $n = 1$:

$$\begin{array}{l|l} \sum_{k=1}^1 k^3 & \left(\frac{1(1+1)}{2}\right)^2 \\ = 1^3 & = \left(\frac{1(1+1)}{2}\right)^2 \\ = 1 & = \left(\frac{2}{2}\right)^2 \\ & = 1^2 \\ & = 1 \end{array}$$

Now, let us assume this works for some fixed, arbitrary $n \in \mathbb{N}$. Consider $n + 1$:

$$\begin{aligned}
 \sum_{k=1}^{n+1} k^3 &= 1^3 + \dots + n^3 + (n+1)^3 \\
 &= \sum_{k=1}^n k^3 + (n+1)^3 \\
 &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
 &= \left(\frac{n^2+n}{2}\right)^2 + (n+1)^3 \\
 &= \frac{n^4 + 2n^3 + n^2}{4} + (n+1)^3 \\
 &= \frac{n^2(n^2 + 2n + 1)}{4} + (n+1)^3 \\
 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
 &= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
 &= \frac{(n+1)^2(n^2 + 4(n+1))}{4} \\
 &= \frac{(n+1)^2(n^2 + 4n + 4)}{4} \\
 &= \frac{(n+1)^2(n+2)^2}{4} \\
 &= \frac{(n+1)^2((n+1)+1)^2}{4} \\
 &= \left(\frac{(n+1)((n+1)+1)}{2}\right)^2
 \end{aligned}$$

Note: this has the proper form. •