Functions: a Special Kind of Relation

You may recall from calculus that a *function* is a rule for uniquely assigning, to every element a in some set A, an element b in some other set B. We will spend the next few days developing a different characterization of these creatures. Strangely enough, we've already developed the tool needed to talk about functions, namely, relations.

Definitions. A *function* f from the set A to the set B (often denoted $f : A \to B$) is a relation $f \subseteq A \times B$ such that for every $a \in A$, there is a *unique* $b \in B$ such that $(a, b) \in f$. We may write the more familiar $f(a) = b$ to mean $(a, b) \in f$, and say that the *value* of f *at the element* a is b. Note that this notation is valid because of our assumption of uniqueness.

Given the scenario described above, A is the *domain* of the function f (often written domain(f)), and B is the *range* (often written range(f)).

Note that the above characterization of functions is likely different from the one with which you're most familiar, in which a function f is thought of as a *rule* for assigning elements of the range to elements of the domain. This new characterization of f (which coincides with what we typically think of as the *graph* of a function) has many advantages. For instance, it's easier to talk about properties like *injectivity* and *surjectivity*, and it's easier to check when two functions are equal. We'll get to these issues after a few

Examples.

1. Let A be the set of English words, and let B be the set of natural numbers. Describe at least two functions with domain A and range B.

2. Write the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ as a relation. Do the same with the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \sin(x)$.

The two functions described in the last example differ in two ways fundamental to the nature of a function.

Definitions. A function $f \subseteq A \times B$ is said to be *one-to-one* (or *injective*) if the following condition is met:

$$(a_1, b), (a_2, b) \in f \Rightarrow a_1 = a_2.$$

That is, if $f(a_1) = f(a_2)$, it must be that $a_1 = a_2$.

Meanwhile, a function $f \subseteq A \times B$ is said to be *onto* (or *surjective*) if the following condition is met:

$$(\forall b \in B)(\exists a \in A)\ (a, b) \in f.$$

That is, every element of B is mapped to by some element of A. If we define the *image* of f (written image(f)) by

$$\text{image}(f) = \{b \in B \mid (\exists a \in A)\ (a, b) \in f\},$$

then f is onto if and only if image(f) = range(f).

Proposition. A function $f \subseteq A \times B$ is one-to-one if and only if the following condition is met:

$$(a_1, b_1), (a_2, b_2) \in f \text{ and } a_1 \neq a_2 \Rightarrow b_1 \neq b_2.$$

Proof. We must prove both directions...

Examples.

1. Let S be a set, and define $f_S : S \to \mathcal{P}(S)$ by $f_S(s) = \{s\}$. Is f_S one-to-one? Is f_S ever onto?

2. Let S be a set, and define $\text{id}_S : S \to S$ by $\text{id}_S(s) = s$ for all $s \in S$. Is f one-to-one? Is it onto?

Definition. A function that is both one-to-one and onto is called a *bijection.*

Examples.

1. Which of the functions considered so far are bijections?

2. If $S = \{1, 2, 3\}$, list *all* bijections from S to itself.

We have developed ways of proving that functions are bijective; many of the techniques we bring from high school algebra come in handy here.

Example. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(T) = \frac{9}{5}T + 32$ is bijective. (*Hint*: carefully show that it is both onto and one-to-one, using the definitions of each of these respective properties...)

Recall that we are always able to define the *inverse* of a given relation: if $R \subseteq S \times T$, then $R^{-1} \subseteq T \times S$ is defined by

$$R^{-1} = \{(t, s) \mid (s, t) \in R\}.$$

Examples.

1. Prove that the inverse f^{-1} of a function f is not always a function, by providing an explicit counterexample.

2. Prove that the inverse of an injection *is* always a function. What are the domain and image of the inverse?

We've all seen formulas for defining functions that rely on rules that sometimes seem to assign more than one value to a given input. For instance, the absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$ can be defined by $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x \leq 0$. If $x = 0$, we have two rules. We're saved by the fact that the two rules *agree* at the point $x = 0$, so that the function is *well-defined*. We say that a function is well-defined if anytime more than one rule can be applied to give the output for a certain input, every "overlapping" rule agrees.

Example. Let S denote the set of names of states in the U.S., and suppose the function $L : S \to \mathbb{N}$ is defined by $L(s) =$ the length of the state name s. Suppose we define the function $\heartsuit$ by $\heartsuit(s) = \sqrt{L(s)}$ if s starts with the letter "O" and $\heartsuit(s) = \frac{L(s)}{2}$ if s has at most 5 letters.

1. What is the domain of $\heartsuit$? The image?

2. Check that $\heartsuit$ is well-defined. (What state names must be checked?)

Let's address one more issue: what does it mean for two functions to be *equal*? Well, let's suppose f and g are both functions from S to T, so that $f, g \subseteq S \times T$. Since we've defined functions to be *sets*, it makes sense to say that $f = g$ if and only if they are equal *as sets*. Let's adopt this definition.

Examples. Are the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ and $g(x) = \frac{x^2}{x}$ equal?

Let $n \in \mathbb{N}$, and let $\mathbb{Z}_n$ denote the set $\{0, 1, 2, ..., n-1\}$. (This notation is very standard. The set $\mathbb{Z}_n$ is called the set of *integers modulo* n; note that $\mathbb{Z}_n$ contains precisely one member of each congruence class of $\equiv_n$.) Define $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $f(x) = x^2$, and $g : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $g(x) = x^3$. Does $f = g$ hold?

What if f and g are defined on $\mathbb{Z}_3$ instead?

We will make much use of functions, particularly bijections, in our discussion of *cardinality*, to which we turn next.