MATH 280, Section 1
Foundations of Mathematics
Fall 2009

Exam 2

Working by yourself, please answer each question below on your own paper. You may use your class notes (but please no other resources!), and you may consult with me (but I promise not to be too helpful!). You may get help neither from each other nor from Math Lab staff nor from any other faculty or staff at UNC Asheville.

Keep in mind that neatness is nice, and that complete sentences are crucial to well-composed writing!

The exam is worth a total of 50 points, and the point value of each question is given with that question. Your completed first draft of the exam is due by 5:00 p.m. on **Friday, November 6th**. Upon receiving your roughly graded exam you will have an opportunity to perform revisions and submit a final version on **Friday, November 13th**.

1. (10 points total) Here we will compute the number of equivalence relations on sets with very small size.

   (a) (2 points) Let $|S| = 1$. Find the number of equivalence relations on S, and compare this to the number of *all* relations on S. (*Hints:* since a relation is a subset of $S \times S$, the second number can be computed using some rules we've already established for the size of products and of powersets...the first number requires that you discover all *equivalence* relations on S. Here it helps to give names to the elements in S: $S = \{a\}$, say.)

   (b) (3 points) Let $|S| = 2$. Find the number of equivalence relations on S, and compare this to the number of *all* relations on S. (*Hint:* see above, with $S = \{a, b\}$, say.)

   (c) (5 points) Let $|S| = 3$. Find the number of equivalence relations on S, and compare this to the number of *all* relations on S. (*Hint:* you know the drill.)

2. (10 points total; 5 points each) Let S be an arbitrary set, and let $R \subseteq S \times S$ be any equivalence relation on R. We define a new relation, $\overline{R}$, on S by

$$\overline{R} = \Big((S \times S) \setminus R\Big) \cup \{(s, s) \mid s \in S\};$$

   that is, we form $\overline{R}$ by taking the complement of R in $S \times S$ and then throwing back in all of the "reflexive" pairs.

   (a) Prove that $\overline{R}$ is always reflexive and symmetric.

   (b) Prove by counterexample that $\overline{R}$ is not always transitive, and thus it is not always an equivalence relation. (*Hint:* consider a set S with 3 elements...)

   (c) **(up to 3 points extra credit)** Prove that if $|S| \in \{1, 2\}$ then $\overline{R}$ *will* always be an equivalence relation.

3. (10 points) Prove carefully why it is that a full house beats a flush and a flush beats a an ordinary straight (*i.e.* not a straight flush) in poker. (For simplicity please assume that aces *cannot* be used as low cards, *only* as high cards, though it turns out that this doesn't affect the result.) Explain *all* of your computations carefully!

4. (10 points) Decide whether the following statement is true or false; if it true, prove it, and if it is false, provide a counterexample to demonstrate that fact:

   For any sets A, B, and C, $(A \setminus B) \times C = (A \times C) \setminus (B \times C)$.

   (*Hint*: approach this problem very carefully, breaking it down as needed using the definitions of "$\setminus$" and "$\times$.")

5. (10 points) Suppose that we live in a city built on a grid system of streets: all roads run either north-south or east-west. Suppose that your favorite restaurant, Salami Slim's Olde Vegan Sandwiche Shoppe, lies 8 blocks west of your apartment, and 4 blocks north. Assuming you can only walk along streets, how many different *shortest* paths can you take from your place to the restaurant? Please explain any computations carefully, justifying them with the proper combinatorial techniques. (*Hint*: is there a way of "encoding" a path symbolically so that you can count the number of paths you may construct?)