MATH 280, Section 1
Foundations of Mathematics
Fall 2009

Exam 1

Working by yourself, please answer each question below on your own paper. You may use your class notes (but please no other resources!), and you may consult with me (but I promise not to be too helpful!). You may get help neither from each other nor from Math Lab staff nor from any other faculty or staff at UNC Asheville.

Keep in mind that neatness is nice, and that complete sentences are crucial to well-composed writing!

The exam is worth a total of 50 points, and the point value of each question is given with that question. Your completed first draft of the exam is due by 5:00 p.m. on **Friday, September 25th**. Upon receiving your roughly graded exam you will have an opportunity to perform revisions and submit a final version on **Friday, October 2nd**.

1. (18 points total) We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *bounded* if and only if there are real numbers m and M such that for all $x \in \mathbb{R}$, $m \leq f(x) \leq M$.

   (a) (4 points) Complete the following definition of a bounded function by filling in the remainder of the definition using *quantifier notation*:

   $f : \mathbb{R} \to \mathbb{R}$ is *bounded* if and only if __________________ .

   (b) (7 points) **True or false?** Every continuous function is bounded. (Please prove your assertion.)

   (c) (7 points) Prove that the sum of two bounded functions is a bounded function.

2. (10 points) Let's define the numbers $\{a_1, a_2, a_3, ...\}$ by letting $a_1 = a_2 = 1$ and

   $$a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-2}}\right)$$

   for all $n \geq 3$. For example,

   $$a_3 = \frac{1}{2}\left(a_2 + \frac{2}{a_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2},$$

   so

   $$a_4 = \frac{1}{2}\left(a_3 + \frac{2}{a_2}\right) = \frac{1}{2} \cdot \frac{7}{2} = \frac{7}{4},$$

   and so forth.

   Prove that for all $n \in \mathbb{N}$, $1 \leq a_n \leq 2$. (*Hint:* Induct. Note that you have to show both inequalities are true at each step, but they may be most clearly shown if you consider them separately.)

3. (14 points total; 7 points each) In class and in a recent homework assignment we showed that if a, b, and c were all odd, then $ax^2 + bx + c$ has no rational roots.

   (a) Assume now that a is even, while b and c are both still odd. What's the strongest claim you can make now about any rational roots of $ax^2 + bx + c$? Prove your claim. (*Hint:* it is possible now to have rational roots, but there are still some restrictions on what they can look like if you examine them *in lowest terms.*)

   (b) Assume now that a and b are both even, while c is odd. What's the strongest claim you can make now about any rational roots of $ax^2 + bx + c$? Prove your claim.

4. (8 points) Find the shortest expression you can which is *logically equivalent to* the following expression:

$$\neg\Big[(\neg Q \vee P) \Rightarrow (P \wedge Q)\Big].$$

   Prove that your expression is indeed equivalent to this one. (You may argue using truth tables, or using already-proven rules for logical equivalence.)