You Can Count on It, III:
Permutations

To end our brief discussion on counting, I'd like to introduce one more concept, this one closely related to combinations.

Definition. Given a set S, a *k-permutation* in the set S is an *ordered list* $(s_1, s_2, ..., s_k)$ of k elements chosen from S.

The important difference between a k-combination and a k-permutation is that the order doesn't matter in the case of a combination: $\{1, 2, 3\}$ and $\{3, 1, 2\}$ both represent the same combination, whereas $(1, 2, 3)$ and $(3, 1, 2)$ are *different* permutations.

Example. Find all k-permutations of the set $\{1, \pi, e\}$, for $0 \leq k \leq 3$.

Now that we've got a formula for the number of k-combinations in a set S with n elements, it's a simple matter to find the number of k-permutations. The key is the following observation:

Lemma. Let $|S| = n$ and suppose $0 \leq k \leq n$. To every k-combination $\{s_1, ..., s_k\}$ in S, there correspond precisely $k!$ k-permutations. Moreover, every k-permutation corresponds to exactly *one* k-combination.

Another way of saying this is that there is a "k!-to-1" function taking permutations to combinations. This is an idea we'll return to once we have a more precise formulation of functions.

Here's some room for us to prove the lemma stated above:

Notice now that this gives us a formula for the number of permutations. From the lemma above, there are exactly $k!$ times as many k-permutations as there are k-combinations. If we let $P(n, k)$ denote the number of k-permutations in a set of size n, we obtain the formula

$$P(n, k) = k! \cdot \binom{n}{k} = k! \cdot \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!}.$$

Examples.

1. Compute the number of 20-letter words that can be formed using the 26 letters of the English alphabet, assuming no letter can be used more than once.

2. Compute the number of ways in which the students in a class with 8 students can be asked to stand in a line.