Section 5.4: The method of u-substitution

Integration is hard. If you're asked to find the antiderivative of a given function, chances are you're going to have your work cut out for you.

The easiest antiderviatives to compute are those that come from obvious differentiation formulas:

Examples. Evaluate each of the following indefinite integrals:

(1) $\int \sin(x)\ dx$

(2) $\int \sec^2(\theta)\ d\theta$

(3) $\int e^r\ dr$

Those of you who go on to Calc II will learn of many powerful techniques to antidifferentiate more complicated functions, but we're going to wrap up the semester with one simple but often applicable method, called the method of u-substitution.

Here's what we're going to do: let's just start with the Chain Rule:

$$\frac{d}{dx}(f \circ g) = f'(g(x))g'(x).$$

If we antidifferentiate both sides of this by-now-familiar formula, the derivative on the lefthand side goes away and we're left with the following:

$$(f \circ g)(x) = \int \frac{d}{dx}(f \circ g)\ dx = \int f'(g(x))g'(x)\ dx = \int f'(g(x))\frac{dg}{dx}\ dx.$$

So what? Let's use the same notation we used when we derived the Chain Rule in the first place, letting $u = g(x)$. The above formula now becomes

$$(f \circ g)(x) = \int f'(u)\frac{du}{dx}\ dx = \int f'(u)\ du.$$

Technically we should justify the way in which we "cancelled" the differentials dx in the last step, but let's suppose this can be done (it can!).

The upshot of these computations is that if we can somehow rewrite a given integral involving x in the form $\int f'(u)\ du$ by a clever "u-substitution" $u = g(x)$, we know that the integral is going to be the composition $f \circ g$.

The Method.

(1) Begin with an integral that looks like $\int f'(g(x))g'(x)\ dx$.
(2) Perform the substitution $u = g(x)$. (In this case, $g'(x)\ dx$ simply becomes du.)
(3) Obtain the integral $\int f'(u)\ du$, which gives the antiderivative $f(u) + C$.
(4) Turn u back into $g(x)$, and celebrate!

Let's just do a slew of examples to see how this works in practice:

Examples.

(1) Compute $\int xe^{x^2}\ dx$. (*Hint:* try $u = x^2$...)

(2) Compute $\int \cos(x)\sqrt{\sin(x)}\ dx$. (What should you use for u here?)

(3) Compute $\int_0^{\sqrt[3]{\pi/4}} x^2 \cos(x^3)\ dx$. (*Hint:* ignore the fact that you have a definite integral at first, and simply solve the corresponding indefinite integral to get an antiderivative, $F(x)$. Then appeal to the FTC to compute $F(b) - F(a)$...)

(4) Compute $\int_0^1 x\sqrt{x^2+1}\ dx$.

By the way, if you don't do the right substitution at first, don't worry! You're not going to break the integral. Some of these guys are difficult to solve, and the more you try different methods as you're first learning how to solve them, the more proficient you'll get at solving them later on.