Sections 5.2 and 5.7: The Fundamental Theorem of Calculus and indefinite integrals

By now we've talked a bit about *definition integrals*. Recall that the definite integral of f on the interval $[a, b]$ is given by

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

for $\Delta x = \frac{b-a}{n}$. If f is always positive, it represents the area underneath the graph of f on the interval $[a, b]$.

Note. If f is *not* always positive, we have to be more careful in our interpretation; the integral $\int_a^b f(x)\ dx$ will in general represent *net* area, which is the area under the ____________ part of the function minus the area under the ____________ part. Here's room for an illustrative example:

Difficulty. Using the definition to evaluate $\int_a^b f(x)\ dx$ for even relatively simple functions f is incredibly difficult. How in the heck are we going to compute these definite integrals quickly?

Some of the values we've computed already hint at an answer: we've already noticed

$$\int_a^b x\ dx = \frac{1}{2}(b^2 - a^2) \quad \text{and} \quad \int_a^b x^2\ dx = \frac{1}{3}(b^3 - a^3).$$

What formula do you suppose will hold in general?

$$\int_a^b x^n\ dx =$$

Observation. Notice that when we take the derivative of a function, we're essentially concerned with rates of change, which involves finding ____________ : the derivative of a function represents the amount of change in the function with respect to change in its independent variable.

When we compute integrals, on the other hand, we're computing line(1,0)50 . Thus if we begin with a function F and find its derivative, f, it shouldn't be all that surprising that a definite integral involving f has something to do with F. This turns out to be the case: the precise relationship between a function F and its derivative f is the content of the most important theorem in all of calculus:

The Fundamental Theorem of Calculus (FTC). Suppose that f is a continuous function on the interval $[a, b]$.

(1) If we define F by $F(x) = \int_a^x f(t)\ dt$ for all x in $[a, b]$, then $\frac{dF}{dx} = f$.

(2) If F is any function satisfying $\frac{dF}{dx} = f$, then $\int_a^b f(x)\ dx = F(b) - F(a)$.

Notes.

(1) Part (1) of the FTC essentially says that the derivative of an integral of a function gives the original function back. Note also the location of the variable x in the definition of $F(x)$: the "t" is called a "dummy variable" since it's just a placeholder and disappears once the integration is performed. The function F is really just giving us the area under the function f starting at a and going to some *variable* point x:

(2) Part (2) of the FTC essentially says that the integral of the derivative of the function is directly related to the original function.

(3) Parts (1) and (2) together show that differentiation and integration are ____________ processes to one another!

The best way to understand the FTC is to play with a bunch of

Examples. In each case below, use the FTC to compute the given definite integral, in each case finding a suitable F to start, and then using this to compute $F(b) - F(a)$:

(1) $\int_0^5 x^3\ dx$

(2) $\int_{-\frac{\pi}{4}}^{\pi} \cos(t)\ dt$

(3) $\int_2^e \frac{1}{x}\ dx$

(4) $\int_{-1}^2 3e^x\ dx$

Obviously it's a useful skill to be able to find the "inverse derivative" of a given function, since the FTC then tells us how to use it to solve definite integrals.

Definition. If f is a function, then any function F such that $\frac{d}{dx}F(x) = f(x)$ is called an ________________ of f. The most general ________________ of f is given by the ____________ *integral* of f, and is denoted by $\int f(x)\ dx$.

Examples. Find the definite integral of each of the following functions.

(1) $f(x) = x^4$

(2) $f(x) = x^n$, for $n \neq -1$

(3) $f(x) = \frac{1}{x}$

(4) $f(x) = \sec^2(x)$

(5) $f(x) = e^x$

Various applications (as in the example below!) will require the use of a *particular* antiderivative, but our general formula above gives us an infinite *family* of such functions!

Note. Since by now we have a copious list of derivatives, we also have a copious list of antiderivatives, including a few general rules, like

$$\int f(x) \pm g(x)\ dx = \int f(x)\ dx \pm \int g(x)\ dx.$$

Your homework asks you to supply several basic examples.

Let's close with one more

Example. Suppose that a watermelon is dropped from a plane flying at an altitude of 1000 meters. The velocity of the watermelon is given by $v(t) = -9.8t$ m/s, where the negative sign indicates downward motion. What is the altitude of the watermelon at a given time t? At $t = 5$ seconds? $t = 10$ seconds?

Hint: distance is the antiderivative of velocity...but *which* antiderivative should you use?

Homework. None of these problems will be graded, but you ought to give them a shot for practice, especially if you intend on continuing to Calc II!

(1) Find the indefinite integral of each of the following functions.
  (a) $f(x) = 3x^2 - 5x + 1$
  (b) $f(x) = 0$
  (c) $f(x) = \pi$
  (d) $f(x) = \cos(x)$
  (e) $f(x) = \sin(x)$
  (f) $f(x) = \cot^2(x)$
  (g) $f(x) = \sec(x)\tan(x)$

(2) Evaluate each of the following definite integrals, using the Fundamental Theorem of Calculus.
  (a) $\int_0^1 4x^5 - 3x^2 - 7x + 2\ dx$
  (b) $\int_{-\pi}^{\pi} 3\cos(x)$
  (c) $\int_0^{\frac{\pi}{2}} \sec^2(x)\ dx$
  (d) $\int_0^1 e^x\ dx$
  (e) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin(x) + x - 1\ dx$

(3) A plate is cut from metal sheeting that weighs 2 g/cm^2. If the plate is shaped like the region contained beneath the function $y = \sqrt[3]{x}$ from $x = 0$ to $x = 27$, find the mass of the plate. (*Hint:* find the area of the plate, and use this to find the mass, knowing the given density.)

(4) A ball is thrown upward with an initial velocity of 10 m/s, and an altitude of 0 m. The velocity of the object at time t is given by $v(t) = 10 - 9.8t$.
  (a) Find the altitude of the ball at a given time t.
  (b) The ball reaches its maximum altitude when its velocity is 0. Find the time t at which this happens.
  (c) Use (b) to find the maximum altitude the ball obtains.