MATH 191, SECTION 2
CALCULUS I
SPRING 2009

Sections 5.2 and 5.3: Riemann sums and definite integrals

Our goal now is to better understand the computation of areas as begun in the last set of notes...and to understand how it relates to the calculus we've done up to now.

At present the connection is a tenuous one: we use a limit (a sort of infinite process, if you will) to compute derivatives, and we use a limit (another infinite process) to compute area. Surely there's a deeper connection than this simple observation.

Our next step in the right direction is to generalize the sort of "infinite sum" we started to talk about in our last class when we were estimating the area under $y = x^2$ from $x = 0$ to $x = 1$.

Definition. Let f be a continuous function on some interval $[a, b]$. Let n be a fixed positive integer, and divide $[a, b]$ into n subintervals, $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$ of equal length, $\Delta x = \frac{b-a}{n}$. (Here $x_0 = a$ and $x_n = b$.) A ____________ *sum* is any sum of the form

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x,$$

where x_i^* is any number in the subinterval $[x_{i-1}, x_i]$.

Accompanying diagram. This is one of those cases in which a picture is worth a thousand (or at least 54, the number of words and expressions in the preceding definition) words. Here's room for a graph illustrating the definition above:

A note on notation. The "$\sum$" appearing in the above definition is called a __________ sum, and is simply used as shorthand for writing sums of indexed terms. That is, if we're given a list of numbers we'd like to add together, $a_1, a_2, ..., a_n$, we can write our list compactly by referring to the terms individually by their ____________ (singular: __________), the little subscripted numbers that tell us where in the list each term lies. The ____________ now allow us to write the sum of the terms compactly as well:

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

the "$i = 1$" merely tells us where to start adding, and the "n" at the top of the "$\sum$" tells where to stop.

The sums we were considering in our last class were special cases of Riemann sums, in which we always chose the *right endpoint* x_i of each subinterval $[x_{i-1}, x_i]$ as our "test point" x_i^*. We could just as easily have chosen the left endpoint x_{i-1} of each subinterval to be our x_i^*.

Example. Let $f(x) = e^x$ and let $[a, b] = [0, 2]$.

(1) Find the value of the Riemann sum with $n = 4$ that uses $x_i^* = x_i$ (the right endpoint) consistently. Draw a picture that illustrates this Riemann sum.

(2) Now find the value of the Riemann sum with $n = 4$ that uses $x_i^* = x_{i-1}$ consistently. Draw a picture illustrating this Riemann sum.

(3) Which of these sums *over*estimates the area under the graph of f? Which *under*estimates the area? Why?

Now let's let n become increasingly large, so the n rectangles into which we divide our interval become increasingly skinny, and all of the points within them get crammed together. Therefore, since f is continuous, it really doesn't matter which point in $[x_{i-1}, x_i]$ we choose as our test point x_i^*:

Fact. The value of the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

does not depend on the choice of the test points x_i^*, as long as $x_i^* \in [x_{i-1}, x_i]$.

This fact is *marvelous!* It means we can pick left endpoints, right endpoints, *midpoints*, whichever points are most convenient, and we'll get the same answer. This "rigidity" in the formula means we can make the following

Definition. Let f be a continuous function on the interval $[a, b]$, and divide $[a, b]$ into subintervals $[x_{i-1}, x_i]$ of length $\Delta x = \frac{b-a}{n}$ as before. Then the *definite* ____________ of f on the interval $[a, b]$ is defined by

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

for *any* choice of x_i^* in $[x_{i-1}, x_i]$.

Notation. The "$\int$" is reminiscent of an elongated 'S,' referring to "*s*um." This is Leibniz's notation. The values a and b are respectively called the *upper* and *lower* _________ of the ____________ . The "dx" should remind you of derivatives...as we'll see there's a reason for this notation!

Interpretation. Note that if $f(x)$ is always positive, all the definite ____________ is telling us, geometrically, is the area under the graph of f on the interval $[a, b]$, since the definition comes from our formula for the area.

Examples.

1. What's $\int_0^1 x^2\ dx$?

2. Compute $\int_2^3 x\ dx$ carefully, using the fact that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$.

   Why is this last computation not at all surprising?

3. Compute $\int_{-3}^0 \sqrt{9-x^2}\ dx$ by interpreting the integral geometrically.

The definition of $\int_a^b f(x)\ dx$ as a limit of sums, prior computation, and straightforward geometric interpretation allow us to use our familiar limit rules to establish some useful facts about definite integrals:

Properties of $\int_a^b f(x)\ dx$. Let f and g be continuous functions on $[a, b]$ and let c be any constant.

(1) $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$
(2) $\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
(3) $\int_a^b f(x) - g(x)\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$
(4) $\int_a^b c\ dx = c(b - a)$
(5) $\int_a^b x\ dx = \frac{1}{2}(b^2 - a^2)$
(6) $\int_a^b x^2\ dx = \frac{1}{3}(b^3 - a^3)$

Property (5) can be checked using the definition or by interpreting the integral $\int_a^b x\ dx$ as the area of a triangle, and Property (6) can be checked through computations much like those we did in our last class.

Example. Use the above properties to compute $\int_{-1}^3 2x^2 - 4x + 5\ dx$.

Homework. The following exercises are due on *Friday, May 1st*, and are the ***last*** graded homework problems of the semester! (I will assign problems for the few topics we'll talk about after those in this handout, but you will not be required to submit them for a grade. I will, however, provide you with feedback if you would like it.)

(1) Draw a graph illustrating the Riemann sum $\sum_{i=1}^4 \sqrt{x_i^*}\Delta x$ on the interval $[0, 2]$, in which x_i^* is always taken to be the right endpoint, x_i, of the subinterval $[x_{i-1}, x_i]$. (What's f here? What's n? Thus, what is Δx? What are a and b?)
(2) Draw a graph illustrating the Riemann sum with $n = 4$ and x_i^* chosen as the left endpoint, x_{i-1}, for the function $f(x) = x^3$ on the interval $[-1, 1]$.
(3) Use the properties of the definite integral, and geometry, if needed, to evaluate each of the following integrals:
    (a) $\int_{-5}^8 2\ dx$
    (b) $\int_3^6 7x^2 - 3x + 1\ dx$
    (c) $\int_0^1 4x - 2 + 5\sqrt{1 - x^2}\ dx$
    (d) $\int_{-4}^4 cx\ dx$, where c is any constant.