Section 5.1: Area

If you ask me, *this* is where calculus *really* starts to get fun! It all begins with a

Problem. Find the area underneath the parabola $y = x^2$ (and above the x-axis) from $x = 0$ to $x = 1$.

Here's some space to draw a graph illustrating our goal:

Before we set about finding the answer, let's figure out why this is a nontrivial problem in the first place, by comparing it with easier problems.

Examples.

(1) Find the area underneath the graph of $y = 3x + 5$ between $x = -1$ and $x = 4$.

You should be able to draw the graph of the function indicated and use geometry to solve this problem. (Recall that the area of a triangle with height h and base b is given by $A = \frac{1}{2}bh$.) Here's some room:

(2) Find the area underneath the graph of the function $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$.

This is really another geometry problem. Draw a graph below and find the area using a well-known formula:

Perhaps you see now why it is that the first problem given to us is hard: can you think of a way to fall back on geometry to make our life easier?

Your answer to the previous question is almost certainly "no": there really is no nice geometry here. We've got to try something different.

To solve the first example above you likely made use of a rectangle, a figure whose area is easy to compute. Suppose we use a few rectangles to *approximate* the area underneath the parabola $y = x^2$? Draw two different graphs below; on the left, indicate how you can use *two* rectangles to get an overestimate for the area underneath the parabola, and on the right indicate how you can use *four* rectangles to do the same:

As you just saw, it's not too hard to compute the areas of the small number of rectangles above. Can we come up with a more *general* means of doing this, for a larger number of rectangles?

Let's suppose we'd like an approximation using n rectangles, for some positive integer n; that means each rectangle has to be $\Delta x = \frac{1}{n}$ units wide. Draw a picture below illustrating this set-up:

Now to compute these rectangles' total area. Note that the right edge of the ith rectangle over lies at the position $i \cdot \frac{1}{n} = \frac{i}{n}$, and that this point is the one that's used to determine how high the ith rectangle is.

In the space below, you should be able to write a formula for the area of the ith rectangle, and a formula for the total area, obtained by adding all of the rectangles' areas up:

Now factor out as many common terms as you can. What are you left with?

Hmmm...we'd be somewhere if we could determine a nice formula for $1+4+9+\cdots+n^2$. Fortunately, such a formula is known:

Fact. The sum of the first n squares, $1^2 + 2^2 + 3^2 + \cdots + n^2$, is given by $\frac{n(n+1)(2n+1)}{6}$.

This fact is similar to Karl Friedrich Gausas's observation that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, which he noticed when he was seven years old!

Now you can use the above "Fact" to get rid of the sum in your formula from before. Do this, and simplify by canceling as much as you can:

Wow! We've got a fairly simple formula for the area included in the n rectangles we drew. But how do we get an *exact* formula for the area we're after from this?

You may have noticed that as we increase the number of rectangles we use, our approximation becomes better and better. This suggests that we use an "infinite" number of rectangles. Since we can't actually use infinitely many rectangles (they'd have width 0!), we do the next best thing and take a ____________ !

Finish it off, using the appropriate ____________ to find the exact value of the area underneath $y = x^2$ from $x = 0$ to $x = 1$:

In the remaining classes of the semester we'll learn how we can compute areas of yet more complicated regions using the aptly-named method of ________________ , an inverse process to differentiation.

Homework. The following exercises are due on *Friday, May 1st.*

(1) Find the *exact* area under the graph of the given function (and above the x-axis), between the given values of x. (*Hint:* a few of these can be done using geometry; the others require computations like those we did above with $y = x^2$.)
  (a) $y = -2x + 7$ between $x = -3$ and $x = 3$
  (b) $y = \pi x + 6$ between $x = 2$ and $x = e$
  (c) $y = 2 - \sqrt{1 - x^2}$ between $x = 0$ and $x = 1$ (*Hint:* draw the graph of this function and divide the region of interest into two pieces...)
  (d) $y = \frac{1}{2}x^2$ between $x = 0$ and $x = 1$ (*Hint:* use n rectangles, each of width $\frac{1}{n}$, just as before.)
  (e) $y = x^2$ between $x = 1$ and $x = 2$ (*Hint:* use n rectangles, each of width $\frac{1}{n}$, as before, and be careful in computing the height of the ith rectangle! You will want to use Gauss's formula for the sum of the first n numbers at a certain point.)