Sections 4.3 and 4.4: Inverse functions and their derivatives, including logs and trig!

It's time for us to find our last differentiation rules. We now focus on the *inverses* of functions. To understand inverses, though, we have to recall another

Definition. Recall that a function f is said to be a ____________ if any two different inputs for f give different outputs; that is,

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

We can test whether or not a function is ____________ using the

____________ **Line Test.** A function f is ____________ if and only if every ____________ passes through the graph of f at most once.

Examples. Demonstrate that $f(x) = x^3$ is ____________ but $g(x) = x^2$ is not, both algebraically and using the ____________ Line Test.

Now what are we going to do with these one-to-one functions?

Definition. If f is a one-to-one function, its *inverse*, written f^{-1}, is defined by

$$y = f^{-1}(x) \Leftrightarrow x = f(y).$$

In this case we get *cancellation rules*:

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(y)) = y$$

whenever these compositions make sense.

Examples.

(1) The one-to-one function $f(x) = x^3$ has what inverse? Graph them both on the same axes below:

(2) The function $f(x) = e^x$ has what inverse? Graph them both on the same axes below:

(3) The function $f(x) = x^2$ is only one-to-one if we *restrict* its domain to $[0, \infty)$. In this case, what's f^{-1}? Graph both the function f and its inverse on the same axes below.

The examples above suggest the following fact:

Observation. If f is a one-to-one function, the graph of the inverse $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ over the line $y = x$. Moreover, the domain of the inverse is the range of the original function, and vice versa.

This shouldn't be surprising: after all, the taking the inverse of a function merely exchanges the roles of x and y.

But what about derivatives? Let's use the Chain Rule to figure out how to differentiate $f^{-1}(x)$. Start with $y = f^{-1}(x)$. Thus $x = f(y)$. Differentiating this equality implicitly, what do you get?

Now, just as we did with implicit differentiation before, solve for $\frac{dy}{dx}$, since that's really $(f^{-1})'(x)$. What do you get? Write your answer proudly below and circle it; it'll be useful!

Specific examples. Applying our newfound method for finding the derivative of an inverse function, we can now obtain the following formulas:

(1) Let $y = \ln(x)$, so $x = e^y$. Here $f(y) = e^y$. What formula do we obtain for $\frac{dy}{dx}$?

(2) Since the function $\sin(x)$ is not one-to-one, we must restrict its domain, much as we did for x^2 before. Let's agree to restrict the domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Draw a graph below showing how on this new domain $\sin(x)$ is indeed one-to-one:

What about the derivative of the *inverse sine* (or ____________) function, $\sin^{-1}(x)$? Let $y = \sin^{-1}(x)$. Now find $\frac{dy}{dx}$, rewriting it in terms of x at the end:

(3) Similarly, we may restrict the function $\cos(x)$ to the domain $[0, \pi]$ in order to ensure that it is one-to-one. Here's some room to draw a graph of the *inverse cosine* (or ____________) function $y = \cos^{-1}(x)$ and come up with a formula for $\frac{dy}{dx}$:

(4) Lastly, let's do $\tan(x)$, restricting its domain to $(-\frac{\pi}{2}, \frac{\pi}{2})$ to make sure it's one-to-one. What derivative do you come up with for the *inverse tangent* (or ____________) function, $\tan^{-1}(x)$?

Homework. The following exercises are due on *Friday, May 1st*, but if you turn them in to me by Friday, April 24th, I'll be sure to grade them and get them back to you before Exam 3.

(1) Find the derivative of each of the following functions.
  (a) $\ln(x^2)$
  (b) $\sin^{-1}(x^2)$
  (c) $\cos(\sin^{-1}(x))$
  (d) $e^{\tan^{-1}(x)}$
  (e) $x\ln(x)$
  (f) $\frac{\ln(x)}{x}$

(2) Just to get some practice with some of the more important inverse functions, give values for the quantities indicated below.
  (a) $\ln(e^2)$
  (b) $e^{\ln(x^2)}$
  (c) $\sin^{-1}(\frac{\sqrt{2}}{2})$
  (d) $\tan^{-1}(-\sqrt{3})$
  (e) $\cos^{-1}(0)$
  (f) $\sin^{-1}(-1)$
  (g) $\sin(\cos^{-1}(\frac{\sqrt{2}}{2}))$

(3) Analyze the function $f(x) = x\ln(x)$ as carefully as you can using the first derivative (to test for extrema and increasing/decreasing behavior), the second derivatives (to test for concavity), and asymptotes.

(4) Repeat (3) with the function $g(t) = \tan^{-1}(t^2)$.

(5) Note that $\lim_{x\to 0^+}\ln(x) = -\infty$ and $\lim_{x\to\infty}\ln(x) = \infty$. You may use these facts without proving them, in order to calculate the following limits. (*Hint:* L'Hôpital the first couple!)
  (a) $\lim_{x\to\infty}\frac{\ln(x)}{x}$
  (b) $\lim_{x\to 0^+}\frac{\ln(x)}{1/x}$
  (c) $\lim_{x\to -4^+}\ln(x+4)$