Section 4.2: Implicit differentiation and related rates of change

Let's dive right in with an

Example. Suppose we want to find $\frac{dy}{dx}$ if we know that $x^2 + y^2 = 1$, but we're too lazy to solve for y to find the right function to differentiate.

Never fear! We are justified in our laziness, for we can differentiate this expression _______________, by merely *thinking* of y as a function of x and using the Chain Rule.

That is, let's think about the lefthand side of the equation above as a function of x and y (say, $F(x, y)$), and think of y in turn as a function of x. Thus the lefthand side is really just a function of x, and by using the Chain Rule to differentiate the y^2, we obtain an equation that can be solved for $\frac{dy}{dx}$:

See how easy?

Note. It's worth noting that if we *were* to solve this equation for y in terms of x, we'd actually obtain *two* functions, the top and bottom of a circle. Here's some room to perform this operation, and to draw a graph illustrating our findings:

The beauty of our _______________ differentiation above is that it gave us the right value for $\frac{dy}{dx}$ no matter which of the two functions we're sitting on!

Here's another

Example. Find $\frac{dy}{dx}$ if $xe^y = \sin(x)$. (*Hint:* as before, just differentiate, being careful to think of y as a function of x and use the Chain Rule...solve the resulting equation for $\frac{dy}{dx}$.)

Now use your formula for $\frac{dy}{dx}$ to find the slope of the tangent line to the graph of the equation above at the point $\left(\frac{\pi}{2}, \ln(\frac{2}{\pi})\right)$.

Implicit differentiation and the Chain Rule also show up in a number of more "applied" problems in which two different rates of change interact (just like in the Chain Rule!), as in this classic problem:

Example. A spherical balloon is being filled with air so that its volume increases by 10 cm^3 per second. How fast is its radius changing when the radius is 20 cm?

As a first step, let's draw a small diagram and write down a formula giving the balloon's volume in terms of its radius:

$$V =$$

Got that? Good. Now note that the quantity we're *given* is $\frac{dV}{dt}$, the rate of change of the volume with respect to time. On the other hand, we're *asked to find* $\frac{dr}{dt}$.

Let's just differentiate the formula we obtained for $V(r)$, with respect to t, and be careful to apply the Chain Rule:

Hey, look! Since we know $\frac{dV}{dt}$, we can plug that in on the left, and since we're given a radius at which we're interested, we can plug that in now, too, and solve for $\frac{dr}{dt}$:

Note. We can only plug in our "static" values, like $r = 20$, at the *very end* of the problem; plugging in sooner "fixes" the problem in time and eliminates the dynamic nature of the rates of change.

Here's another

Classic example. A 3-meter ladder is resting against a wall and its base (resting on a banana peel) slides away from the wall at 0.5 meters per second. How fast is the ladder's top sliding *down* the wall when the top is 2 meters from the ground? 1 meter? 0 meters? (*Hint:* Let x be the distance of the ladder's bottom from the wall, and let y be the height of the top...we're given the rate of change $\frac{d???}{dt}$...)

One more classic. A plane is flying at an altitude of 3 km, on course to fly over our heads. If the plane is in level flight, traveling at 500 km/hr, how fast is the plane nearing the point where we stand when it's flying over a point 4 km away? (*Hint:* definitely draw a diagram, and name some variables. Be very careful about the Chain Rule here!)

Homework. The following exercises are due on *Friday, April 24th.* In most of the following exercises I *strongly* recommend by starting out with a diagram and some notation!

(1) For each expression given below, find $\frac{dy}{dx}$ by implicit differentiation. If a point (x, y) is given, use your derivative to find the equation of the tangent line to the expression's graph at that point. (Remember once you have the slope of the tangent line, the equation of line is easily obtained from the point-slope formula.)
   (a) $x^2y + y^2x = 1$
   (b) $x^2 = \sin(y)$, $(x, y) = (0, \pi)$
   (c) $ye^x = \cos(y)$
   (d) $x^{10} + y^{10} = 2$, $(x, y) = (1, 1)$

(2) As it falls earthward, the surface area of a spherical raindrop is increasing at a rate of 1 mm^2 per second. How fast is the radius of the raindrop increasing when its radius is 2 mm? (*Hint:* the area of a sphere of radius r is $A = 4\pi r^2$.)

(3) A camera is trained on a rocket as it launches. If the camera is 1 km away from the rocket and the rocket rises at 1 km/sec, how fast is the angle the camera's view makes with the ground increasing when the rocket is 1 km off the ground? (*Hint:* find a relationship between the height y of the rocket and the angle θ...)

(4) The *ideal law for gases* states that the temperature T, the pressure p, and the volume V of a fixed quantity of gas obey the relationship $pV = cT$, where c is a positive constant.

(a) Differentiate the gas law with respect to time, t, being careful to use the Product Rule and the Chain Rule where applicable (since every one of T, p, and V could depend on time, the Chain Rule will come up a few times!).

(b) Suppose you know that the temperature of the gas is held fixed (what's $\frac{dT}{dt}$ in this case?), and that the pressure is increasing. What can you say about the rate of change of the volume of the gas, knowing that pressure and volume are always positive? (*Hint:* solve your equation in (a) for $\frac{dV}{dt}$...)

(c) Suppose now that the gas is being compressed, but the pressure is also decreasing. What can you tell about the temperature?