MATH 191, Section 2
Calculus I
Spring 2009

## *Section 3.8: I want my MVT!*

We had a brief encounter earlier with one of the most important theorems from calculus, namely the Intermediate Value Theorem (IVT). You may recall that IVT says the following:

**Intermediate Value Theorem.** Let $f$ be a function that is continuous on the closed interval $[a, b]$. Then for any number $N$ (on the $y$-axis) between $f(a)$ and $f(b)$, there is a number $c$ (on the $x$-axis) on the interval $[a, b]$ such that $f(c) = N$.

Just to refresh your memory, you should draw a picture of the IVT in the space below:

Another important theorem appeared on the same page of your textbook as IV T(page 89, to be precise), but we didn't pay it much attention at the time:

**Extreme Value Theorem (EVT).** Let $f$ be a function that is continuous on the closed interval $[a, b]$. Then there are numbers $c$ and $C$ at which $f$ actually attains its least and greatest values on the interval $[a, b]$. (That is, the function really *does have* maxima and minima on any closed interval.)

**Examples.** In the space below, draw pictures of what could go wrong in EVT if...

(1) ...$f$ is not continuous on $[a, b]$:

(2) ...or $f$ is continuous on a non-closed interval, like $[a, b)$:

Clearly continuous functions behave nicely on closed intervals!

Perhaps the most important of all "value" theorems is the

**Mean Value Theorem (MVT).** Suppose that $f$ is continuous on the interval $[a, b]$ and differentiable on the interval $(a, b)$. Then there exists a number $c$ on $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Let's first take a minute to understand what MVT is telling us.

Geometrically, what does $f'(c)$ tell us about the function $f$ at the point $x = c$?

Geometrically, what does $\frac{f(b)-f(a)}{b-a}$ tell us about the function $f$?

MVT says these values are equal! You should be able to use this information to draw a graph indicating what MVT is telling us below (in fact, draw three or four graphs, side-by-side!):

As a first step towards proving the MVT, let's prove

**Rolle's Theorem.** If $f$ is continuous on the interval $[a, b]$ and differentiable on the interval $(a, b)$ and $f(a) = f(b)$, then there is a point $c$ on $(a, b)$ such that $f'(c) = 0$.

Notice that this is just the special case of the MVT in which the function $f$ has the same value at the endpoints of our interval.

*Proof.* Let $f$ be as given above. There are two possibilities:

**Case 1.** The function is constant on the interval $[a, b]$. In this case, *any* $c$ on the interval $(a, b)$ will do!

**Case 2.** If the function is *not* constant on the interval $[a, b]$, then it either has a maximum or a minimum at some point $c$ in $(a, b)$. Since the function is differentiable at this $c$, the derivative $f'(c)$ exists. What do we *know* it must equal?

And we're done!

If $f$ does not have the same value at the two endpoints, we have to do a little more work: the secant slope we're trying to match with our tangent slope is not horizontal. However, the point we're looking for is still going to be a point at which *something* is maximized.

**Claim.** The point $c$ we're looking for occurs when the distance between the function and the secant line through the endpoints $x = a$ and $x = b$ is maximized.

Here's some room for you to draw a picture indicating this claim:

Let's prove MVT using the above claim.

*Proof of MVT.* The distance between the function $f(x)$ and the secant line $\frac{f(b)-f(a)}{b-a}(x-a)+f(a)$ is

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a).$$

Since $f$ is continuous and differentiable, so is $F$.
Why is this so?

Notice that $F$ satisfies $F(a) = F(b)$. Here's some room to check that fact:

Aha! This means that Rolle's Theorem applies to $F$, enabling us to find a number $c$ such that $F'(c) = 0$. What's this say about $f'(c)$? Here's some room to investigate the consequences of $F'(c)$:

Now that we know that MVT is *true*, let's think about how it might be *useful.*

**Application.** The police are attempting to crack down on speeders driving through a particularly long tunnel where it's difficult to monitor motorist's velocities. If the speed limit through the tunnel is 100 kph (kilometers per hour) and the tunnel is 10 kilometers long, explain how the police could use the MVT to *prove* that if someone enters the tunnel at 3:00 p.m. and exits at 3:03 p.m. breaks the speed limit at some point inside the of the tunnel.

Notice that in the statement of the MVT, we are not told *where* $c$ is, only that it exists. In some cases, we might actually be able to find the value of $c$ that works explicitly:

**Example.** Let $f(x) = x^3 - 2x^2 + 3x$ on the interval $[0, 1]$.

(1) First, explain why the MVT applies in this case.

(2) Now, find the slope of the secant line through the endpoints $x = 0$ and $x = 1$.

(3) Finally, using an explicit formula for $f'$, find the value of $c$ that will ensure that the tangent through $x = c$ will have the same slope as the secant slope you found above. (*Hint*: quadratic formula!)

Here's one more application of the MVT:

**Proposition.** Let $f'(x) = 0$ for all $x$ in some interval $[a, b]$. Then $f$ is constant on the interval $[a, b]$.

*Proof.* Suppose $f'(x) = 0$ for all $x$ on $[a, b]$, and let $c_1$ and $c_2$ be numbers in $[a, b]$, with $c_1 < c_2$. Consider the quotient $\frac{f(c_2)-f(c_2)}{c_2-c_1}$. What does the MVT tell us about this quotient, for some $c$ in $(c_1, c_2)$?

You should be able to finish the proof from there:

**Homework.** The following exercises are due on *Friday, April 17th.*

(1) Let $f(x) = x + \frac{1}{x}$.
  (a) Explain why the MVT does not apply to the function $f$ on the interval $[-1, 1]$.
  (b) Explain why the MVT *does* apply to the function $f$ on the interval $[1, 2]$, and find all numbers $c$ on the interval $(1, 2)$ that make the MVT true.

(2) Find all points $c$ on the interval $(2, 4)$ which make the MVT true, if $f(x) = 2x^3 - 4x^2 + x - 5$. Draw a graph that illustrates your findings.

(3) Police have switched their attention to a longer tunnel with a lower speed limit. The new tunnel is 15 km long and its speed limit is 90 kph. If a driver enters the tunnel at 3:00, what is the *earliest* time at which the driver can leave the tunnel without the police being able to *guarantee* the driver sped at some point in the tunnel? (*Hint*: as in the example completed in class, let $f(t)$ be the distance the driver has gone from the end of the tunnel $t$ hours after 3:00. What value of $t$ makes the secant slope equal 90?)