

Section 3.8, Part II: Takin' it to the Hospital

Let's examine one more application of derivatives before we investigate our last (and most useful!) rule for computing them.

Sometimes when we're computing the limit of a quotient, say $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$, our quotient is continuous at $x = a$, so we can just plug in $x = a$ to find out what the limit is.

Example. Compute the limit $\lim_{t \rightarrow 4} \frac{t^2+1}{2t^2-7t+1}$.

Sometimes the quotient isn't continuous, but has a clear asymptote because the denominator approaches 0 and the numerator approaches some nonzero value.

Example. Compute the limit $\lim_{x \rightarrow 2^-} \frac{x+3}{x-2}$. (Recall that the "2⁻" means that we are to compute the *one-sided* limit, considering only values of x less than 2.)

Other times, however, our function isn't continuous and there isn't an obvious asymptotes.

Examples. Guess each of the following limits:

(1) $\lim_{x \rightarrow 1} \frac{\ln(x)}{x-1}$

(2) $\lim_{\theta \rightarrow 0} \frac{\theta - \sin(\theta)}{1 - \cos(\theta)}$

These limits are both examples of _____ forms of type $\frac{0}{0}$: both the top and bottom of the quotient tend to 0 when we take the limit, making it hard to determine the limit.

There are other indeterminate forms, including both $\frac{\infty}{\infty}$ and $0 \cdot \infty$:

Examples. Guess each of the following limits:

(1) $\lim_{x \rightarrow \infty} \frac{x^2}{e^x}$

(2) $\lim_{x \rightarrow \frac{\pi}{2}^+} (x - \frac{\pi}{2}) \tan(x)$

One way to determine these limits (as you have above) is to graph the functions given and estimate by eyeballing them.

Alternatively, we may use instead the following fact, attributed to Guillaume de L'Hôpital, a late-17th century French mathematician who wrote the first ever calculus textbook:

L'Hôpital's Rule. Suppose that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ are either both 0 or both $\pm\infty$ (you can mix the signs on these infinities). Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

provided this second limit exists. (Note that we may also replace a by either ∞ or $-\infty$; that is, this rule works for limits at infinity as well.)

Note. Instead of proving this theorem, we simply note that it "should" be true by observing that as we zoom in on the graphs of the functions f and g at $x = a$, the tangent lines to these functions more and more closely approximate the functions themselves. That is, the ratio of the functions can be approximated closely by the ratio of the slopes of their tangent lines, which is what L'Hôpital's Rule is really saying.

Examples. Compute each of the following limits using L'Hôpital's Rule.

(1) $\lim_{\theta \rightarrow 0} \frac{\theta - \sin(\theta)}{1 - \cos(\theta)}$

(2) $\lim_{x \rightarrow \infty} \frac{x^2}{e^x}$ (*Note:* maybe you can apply L'Hôpital's Rule more than once?...)

(3) $\lim_{x \rightarrow \frac{\pi}{2}^+} (x - \frac{\pi}{2}) \tan(x)$ (*Note:* maybe you can turn this product into a quotient by rewriting $\tan(x)$...)

(4) $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$ (Does this limit look familiar?)

Before we close, we note an important

Caveat! You can *not* apply L'Hôpital's Rule if you don't have an indeterminate form: you have to *check* to make sure you've got either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before applying the rule. Most mistakes in using L'Hôpital's Rule stem from misapplying it where it doesn't belong.

Example. Compute the limit $\lim_{x \rightarrow 1} \frac{2x^2 - 1}{x^3 + 6x - 5}$.

What value do you get if you *erroneously* apply L'Hôpital's Rule first?

Homework. The following exercises are due on *Friday, April 17th*.

- (1) Compute each of the following limits:
 - (a) $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$
 - (b) $\lim_{\theta \rightarrow 1} \frac{\theta - 1}{\sin(\theta)}$
 - (c) $\lim_{\theta \rightarrow \pi} \frac{\theta - \pi}{\sin(\theta)}$
 - (d) $\lim_{x \rightarrow \infty} \frac{6x^5 - 4x^3 + 2x + 1}{5x^5 - x^2 + 7}$
 - (e) $\lim_{x \rightarrow \infty} \frac{x^3}{e^x}$
 - (f) $\lim_{x \rightarrow \infty} \frac{x^4}{e^x}$
 - (g) $\lim_{x \rightarrow \infty} \frac{x^5}{e^x}$
- (2) Based upon the last three limits in the previous exercise, what do you suppose $\lim_{x \rightarrow \infty} \frac{x^n}{e^x}$ is, for any natural number n ? In your own words, describe what this means about the manner in which x^n and e^x grow as $x \rightarrow \infty$.
- (3) Consider $\lim_{x \rightarrow \infty} \frac{x + \cos(x)}{x + \sin(x)}$.
 - (a) Show that L'Hôpital's Rule doesn't work to find this limit. (*Hint:* $\cos(x)$ and $\sin(x)$ oscillate periodically...does $\lim_{x \rightarrow \infty} \cos(x)$ even exist?)
 - (b) Guess the value of $\lim_{x \rightarrow \infty} \frac{x + \cos(x)}{x + \sin(x)}$ by noting that $x + \cos(x) \approx x$ and $x + \sin(x) \approx x$ when x is large.