MATH 191, Section 2
Calculus I
Spring 2009

Section 3.3: Further properties of graphs

We now understand pretty well how it is that the derivative of a function relates to the graph of the original function.

But you may recall that we *briefly* examined an extension of the ordinary derivative: the ____________ *derivative* of the function $f(x)$, denoted $\frac{d^2 f}{dx^2}(x)$ or ________ , is defined to the be the derivative of the ______________ of f.

Note. We can often continue differentiating, defining the third derivative $f'''(x)$, the fourth derivative $f''''(x) = f^{(4)}(x)$, and so forth. These derivatives have subtle meanings you'll learn more about in Calc II.

Definition. The second derivative of the function f measures the function's *concavity*, or how much it's "bending," either up or down.

Example. Consider the function $f(x) = x^3$. Since $f'(x) = 3x^2$ is always at least 0, the function f is always increasing. However, something interesting *does* happen at the critical point $x = 0$, as you can see in the graph of $f(x)$ given below:

How does the graph change *qualitatively* as we pass the point $x = 0$?

Now go back to your graph above and draw a few tangent lines to the graph, making sure to include at least one on either side of the point $x = 0$.

Regarding these tangent lines:

(1) If $a < 0$, where does the tangent line at $x = a$ lie, relative to the function's graph?

(2) If $a > 0$, where does the tangent line at $x = a$ lie?

It turns out that the second derivative is *exactly* what we need to keep track both of the "bendiness" of the graph, and of the location of those tangent lines:

Definition/Proposition. Let f be a given function.

(1) If $f''(x) > 0$ for all x on an interval (a, b), then f is *concave* _____ on that interval: it "bends upward" and its tangent lines lie _________ the graph of f.
(2) If $f''(x) < 0$ for all x on an interval (a, b), then f is *concave* _____ on that interval: it "bends _________ " and its tangent lines lie _________ the graph of f.

By the way, if f'' changes sign (from negative to positive or *vice versa*) at the point $x = a$, we call that point an _________________ *point* of f.

Looking back at the graph of $f(x) = x^3$, what happens to tangent lines at an inflection point?

Example. Consider $f(x) = x^4 + \frac{4}{3}x^3 - 12x^2$.

(1) Find the intervals on which f is increasing and decreasing, as well as all critical points, maxima, and minima of the function.

(2) Find the intervals on which the function f is concave up, and those on which it is concave down. Are there any inflection points?

Hey, check out the extrema in this example: do you notice any relationship between the location of the extrema and the concavity of the function at those points?

This last observation is no coincidence:

Proposition. Let $f'(a) = 0$. If $f''(a) > 0$, the point $x = a$ is a local ____________ of the function f. If $f''(a) < 0$, the point $x = a$ is a local ____________ of the function f.

Example. Consider the function $f(x) = x^4 - 2x^2$. In the space below, analyze the graph of this function as fully as you can using $f'(x)$ and $f''(x)$.

The second derivative of a function can do a little more for us than tell us about concavity. Remember linear approximations? The tangent line to the graph of f at the point $x = a$ gives an easy (but not always accurate) estimate for $f(x)$ near $x = a$:

$$f(x) \approx f'(a)(x - a) + f(a)$$

when x is near a.

Second derivatives allow us to give a slightly *better* estimate using ____________ instead of lines:

Definition/Proposition. Let f be a given function. Then the ____________ *approximation* to f at the point $x = a$ is given by

$$\frac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a).$$

This function has the same value, first derivative, and second derivative as f at the point $x = a$.

Notice that the this approximation matches the original function's position, increasing/decreasing behavior, *and* concavity: it's no wonder they're so close.

Just for fun, how do you think you would define the *cubic approximation* to f at $x = a$?

Example. Find both the linear and the quadratic approximations to each of the following functions, at the given point.

(1) $f(x) = e^x$ at the point $x = 0$

(2) $g(x) = \sin(x)$ at the point $x = \frac{\pi}{4}$

Homework. The following problems are due on *Friday, March 27th.*

(1) For each of the following functions, find all
   (i) critical points,
   (ii) extrema of any kind,
   (iii) intervals of increase and decrease,
   (iv) intervals of concavity, and
   (v) inflection points.

   Please show all of your work, and note that a graph alone is not sufficient.

   (a) $f(x) = x^5 - 5x^4$
   (b) $g(x) = x^3 e^x$
   (c) $h(x) = \frac{e^x}{x^2}$

(2) Draw the graph of a *single* function f having the following properties (you do *not* have to give a formula for the function):
   (a) a horizontal asymptote at $y = 0$,
   (b) a vertical asymptote at $x = -2$,
   (c) a global minimum at $x = 0$,
   (d) no global maximum,
   (e) its only x-intercept (that is, a root) at $x = 1$, which is also an inflection point of f, and
   (f) a critical point at $x = 2$ that is not an extremum but *is* an inflection point.

(3) Find both the linear approximation and the quadratic approximation to the function $f(x) = \cos(x)$ at $a = 0$.

(4) Find both the linear approximation and the quadratic approximation to the function $f(x) = \sqrt{x}$ at $a = 100$. Having found these approximations, use both of them to estimate $f(99) = \sqrt{99}$. How far off are your answers from the real value, which is roughly 9.9499874371?

(5) The *quartic approximation* to the function f at the point $x = a$ is given by

$$\frac{1}{24}f^{(4)}(a)(x-a)^4 + \frac{1}{6}f'''(a)(x-a)^3 + \frac{1}{2}f''(a)(x-a) + f'(a)(x-a) + f(a).$$

   Prove that at $x = a$ this function has the same value and first, second, third, and fourth derivatives as the original function f.