Section 3.2: Maxima and Minima

Many applications of the derivative involve "optimization" in some sense:

(1) Among all possible sale prices for an automobile, find the price that will yield the *greatest* profit.
(2) Among all possible energy pathways for a particle undergoing radioactive decay, find the pathway that will yield the *greatest* energy release.
(3) Find the level of light at which the population of a harmful bacterium grows *least* quickly.
(4) Among all possible shapes enclosing a given (fixed) volume, find the shape with the *least* surface area.

I've italicized the crucial word in each of the above examples: in every case we're looking for the place at which some function achieves its greatest or least value. That is, we're looking for *maxima* and ____________ . (Collectively, these points are called the ______________ of the function.)

Although we've yet to put it alone in the spotlight, we've already seen a means of locating the values at which a given function reaches its extrema. Here's a key observation:

Proposition. Let f be a function. If $f'(x) > 0$ for all x in the interval (a, b), then f is ________________ on the interval (a, b). If $f'(x) < 0$ for all x in (a, b), then f is ________________ on that interval.

Examples.

(1) Find the interval(s) on which the function $f(x) = x^3 + 3x^2 - 24x$ is increasing. Find the interval(s) on which f is decreasing.

(2) Now find the intervals of increase and decrease for the function $g(x) = \frac{1}{x+1}$.

(3) When is the function $h(x) = 1 + \sin(x)$ increasing?

We're now in a position to talk about extrema. In the first example above (not to mention in previous examples from class) you surely noticed the following fact:

Provided f is differentiable, f has an extremum at a value x at which is transitions from increasing to decreasing, or *vice versa*.

More precisely, if f goes from increasing to decreasing at x, f has a ________________ at x. If f goes from decreasing to increasing at x, f has a ________________ at x. Such extrema are called ________________ *extrema* because we cannot be assured that the *overall* extreme value is reached at x, only that the extreme value for that "area" is reached at x.

Restating the above observation in terms of f' instead of f itself, we obtain the following important fact:

Proposition. If f is differentiable and f has a local extremum at x, then $f'(x) = 0$.

Thus if we want to hunt extrema, we want to look for x such that $f'(x) = 0$. Before we go on some brief extremum safaris, you should be able to come up with an example of a function f which has an extremum at some point x, but such that $f'(x)$ does not exist (and therefore cannot be 0):

Examples. For each function given below, find the local extrema of the function.

(1) $f(x) = 4x^3 - 3x^4$ (**Note:** does every x satisfying $f'(x) = 0$ give an extremum here?)

(2) $g(x) = e^x - x$

(3) $h(x) = x + \frac{1}{x}$

In the above examples you should note that sometimes a local extremum is actually an _____________ (or *global*) *extremum*, the point at which a function obtains its "overall" greatest or least value. Other times, there is no global extremum, because the function becomes arbitrarily large in the negative or positive direction.

Examples. List the global extrema for each of the above examples, including the value the function obtains at those points.

The points at which the derivative of a function either equals 0 or does not exist (if there are such points!), as well as the endpoints of the functions domain (if they exist!) are called the function's _____________ *points*. You should become adept at finding these points.

Equipped with a good understanding of the behavior of a given function at its critical points, we can now consider a couple of

Applications.

(1) A rectangular box (with a lid) must be made out of sheet metal whose cost is directly proportional to its area. The box must be made in such a fashion that its volume is 1000 cubic centimeters, and so that its width is twice its length. Find the dimensions of the box that can be most cheaply manufactured.

(2) Now we need to make sheet-metal cylindrical cans (again with a lid) holding 1000 cm^3. Find the radius for which the corresponding can can be made as cheaply as possible. (*Hint:* first find formulas for the volume and surface area A of the cylindrical can, involving both radius r and height h. Our constraints then allow you to solve for h in terms of r alone, and so to optimize A as a function of r.)

(3) Market research for our company, *Melons R Us*, indicates that the cost of producing and distributing melons is given by the formula $C(m) = 14 + e^m - 5m$ cents per melon, where m indicates millions of melons sold. Our company can only manage to sell up to 3 million melons, so the domain we wish to consider is $[0, 3]$. Your task: find the production level (that is, the value of m) at which the production cost per melon is minimized.

Homework. The following problems are due on *Friday, March 20th.*

(1) Find all critical points, local extrema, and global extrema of each of the following functions. (For each, you should clearly indicate which points are of which type! Please show all of your work, too, and note that a graph alone is not sufficient.)
   (a) $f(x) = x^4 + x^3 - 2x^2$
   (b) $g(x) = \sin(x)$ on the interval $[-\frac{\pi}{2}, \frac{3\pi}{2}]$
   (c) $h(x) = -x^2 + 6x - 2$
   (d) $k(t) = \frac{1}{t} - \frac{1}{t^2}$ on the interval $[1, 3]$
   (e) $m(x) = \frac{2t^2-3t+2}{t^2+2}$ on the interval $[0, 5]$

(2) Show that of all rectangles with fixed area 100, the 10×10 square has the least perimeter. (*Hint:* minimize the perimeter P of a rectangle with area 100 after finding a formula for P in terms of, say, the width w of the rectangle.)

(3) Suppose that we're manufacturing cylindrical cans, as in the second application considered in class, but now the top and bottom of our cans are to be made of a slightly more expensive material that costs twice as much as the material used to make the sides of the cans. Specifically, the cost of the "side" material is 0.1 cents per cm^2, while the cost of the "top and bottom" material is 0.2 cents per cm^2. Find the dimensions of the can whose manufacture minimizes cost, still assuming that the volume must be 1000 cm^3.

(4) Read the article located at

http://www.sciencenewsforkids.org/pages/puzzlezone/muse/muse0104.asp

and explain as carefully as you can how the dog Elvis is essentially doing calculus. (Specifically, you should describe how we might set up Elvis's mental calculations formally!)