MATH 191, Section 2
Calculus I
Spring 2009

Section 3.1: Linearizations

We've already done a little bit with ____________ approximations, when we used the ____________ line to the graph of a function at $x = a$ in order to approximate the function's value near a.

Since the slope of the tangent line at $x = a$ is given by ________ , we have the following formula for the tangent line:

$$y - f(a) = f'(a)(x - a).$$

Our claim is that the y value given by this line is close to $f(x)$ if x is close to a. Thus we have

$$f(x) \approx f'(a)(x - a) + f(a).$$

Examples. Find the tangent line approximation for the function $f(x) = x^2$ at the point $x = 1$, and use it to estimate the value of $f(1.5)$.

Now do the same for the function $g(x) = x^3$. Which approximation is better?

We can rewrite our approximation formula $f(x) \approx f'(a)(x-a)+f(a)$ by focusing not on x but rather on the difference between x and a. This difference is usually represented, as in our limit formulas for the derivative, by Δx or h.

Letting $\Delta x = x - a$, we have x ______ , and we thus obtain the new formula

$$f(a+\Delta x) \approx f'(a)\Delta x + f(a).$$

Examples. Compute an estimate for the values $(1+h)^n$ when h is *very* small. (*Hint:* let $f(x)=x^n$ and $a=1$.)

If we let $h=-x$ in the formulas you found above, we get a nice linear approximation for $f(x) = \frac{1}{\sqrt{1-x}}$ for small values of x:

Application. Let's do some relativistic mechanics! Relativity predicts that objects get heavier as they get faster, until at the speed of light, any object will have infinite (theoretical) mass.

Supposing that m_0 is the ______ *mass* of a given object, we can compute the mass of the object at velocity v by the formula

$$m(v) = \frac{m_0}{\sqrt{1-(v/c)^2}},$$

where $c = 299,800,000$ meters per second is the speed of light in a vacuum.

Note. Both m_0 and c are constants here!

Step 1. Use the estimate for $\frac{1}{\sqrt{1-x}}$ in the second example above to get an estimate for $\frac{1}{\sqrt{1-(v/c)^2}}$. (*Hint:* just use $x = \cdots$?)

Step 2. Now find an estimate for $m(v)$ using the formula you just found.

Step 3. Let's denote the change in mass (from rest mass to "relativistic mass") by Δm. What *is* Δm, roughly, according to our approximation?

Step 4. Physics dictates that the ______________ *energy*, let's call it E, of an object is given by $\frac{1}{2}m_0v^2$. How can we write this in terms of Δm?

Step 5. Does this formula look at all familiar? Where have you heard it before?

We just derived some highly nontrivial statement about modern physics, just by using linear approximations!

The homework will ask you to compute a few more estimates using linear approximations; they're very useful.

Homework. The following homework is due at 5:00 p.m. on *Friday, March 20th.*

1. Find the linear approximation to the function $f(x) = \sin(x)$ at the point $a = \frac{\pi}{2}$.
2. Consider the function $f(x) = \sqrt{x}$.
   (a) Find the linear approximation to the function $f(x)$ at the point $a = 100$.
   (b) Use your formula from (a) to approximate $\sqrt{101}$, knowing that $f(100) = 10$. How does your estimate compare with the actual value of $\sqrt{101}$? Were you able to accomplish this estimate without a calculator?
3. Following the same procedure as outlined in (1), use the function $f(x) = \sqrt[3]{x}$ and the value $a = 1000$ to approximate $\sqrt[3]{999}$. How does your estimate compare with the actual value of $\sqrt[3]{999}$?
4. The volume of a sphere is given by $V(r) = \frac{4}{3}\pi r^3$, where r is the sphere's radius.
   (a) Find the linear approximation to V at the point $r = 10$.
   (b) Use the formula from (a) to estimate the volume of a sphere of radius 11. How does your estimate compare with the actual value of $V(11)$?