Section 2.7: Continuing with continuity

Recall that in our computation of limits in the last couple of classes, we casually posited the following

Definition. Suppose that a function f satisfies $\lim_{x\to a} f(x) = f(a)$. Then the function is said to be ______________ at the point $x = a$.

No fewer than *three* claims about the function f are being made here, every one of which must hold if f deserves to be called ______________ at a:

1. $\lim_{x\to a} f(x)$ must be ______________ ,
2. $f(a)$ is defined, and
3. these two quantities must be equal!

There are a number of ways a function can *fail* to be continuous at $x = a$; if this happens we say that f has a ________________ at $x = a$. Here are some different scenarios:

1. **Jump discontinuities.** The step function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0 \end{cases}$$

has such a discontinuity. Graph the function to see why this terminology makes sense:

2. ______________ **discontinuities.** The piecewise function

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x^2 & \text{if } x \neq 0 \end{cases}$$

has this sort of discontinuity.

If you graph this function you can see how redefining the function at a single point would "__________ " the discontinuity, giving us the term we use to describe it:

Note that the first function we considered could not have been "fixed" so easily, because of the presence of the *one-sided* limits.

3. **Vertical asymptotes (______________ discontinuities).** The function $f(x) = \frac{1}{x}$ has such a discontinuity:

   Your text notes that if we consider the one-sided limits in this case and allow ourselves to consider infinite values as limits, we can make the following claims:

$$\lim_{x\to 0^-} \frac{1}{x} = \_\_\_\_\_ \qquad \text{and} \qquad \lim_{x\to 0^+} \frac{1}{x} = \_\_\_\_\_ \quad .$$

4. **Oscillatory discontinuities.** Your text gives the curious case of $f(x) = \sin(1/x)$, whose graph *Mathematica* can help us understand:

The behavior of this function near $x = 0$ is positively *insane*! There's no way to remove this discontinuity.

Just to make sure you're understanding these different sorts of pathological behavior, try the following

Example. Draw the graph of a *single* function that illustrates every one of the sorts of discontinuity given above:

If a function is continuous at *every point* in its domain, we say simply that the function is *continuous*, with no extra "at $x = a$" to modify it.

Examples. The following sorts of functions are continuous:

1. Polynomials
2. Radical functions
3. Trigonometric functions
4. Exponential functions
5. Logarithmic functions

Note that rational functions will be continuous everywhere except where their denominators are 0: at those points we'd have either vertical asymptotes or "holes" in the graph.

Example. Graph the function $f(x) = \frac{x^2+2x-3}{x^2-3x+2}$. What sort of discontinuities does it have, and where?

It turns out that the functions we'll deal with most are continuous functions; in fact, they'll usually satisfy an even stronger property, they'll be ________________ :

Definition. A function f is said to be ________________ at the point $x = a$ if $f'(a)$ can be defined.

Examples. Almost all of the sorts of continuous functions described above are also differentiable everywhere they're defined. The only tricky ones are some of the radical functions.

Example. Where is the function $f(x) = \sqrt[3]{x} = x^{1/3}$ continuous? Where is it differentiable?

The following fact sometimes proves useful:

Theorem. If a function is ________________ at the point $x = a$, then it is ____________ at the point $x = a$.

The *converse* is not true:

Example. What's true about the function $f(x) = x^{1/3}$ considered above?

Your text wraps up this section (and this chapter!) with a couple of observations about continuous functions, the most important of which is the following:

Intermediate Value Theorem (IVT). Suppose that $f(x)$ is continuous at every point in some closed interval $[a, b]$, and that F is some number in between $f(a)$ and $f(b)$. Then there is a number c on the interval $[a, b]$ such that $f(c) = F$.

This theorem sounds complicated, but the picture's really pretty obvious. Here's some space to draw a picture of what's going on:

Examples. To see that we *need* the continuity assumption to make things work in the IVT, let's consider the step function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

If we let $F = \frac{1}{2}$ and consider the interval $[-1, 1]$, can we make the claim that the IVT makes? Draw a picture below and see for yourself!:

Homework. The following homework is due at 5:00 p.m. on *Friday, March 6th.*

1. For each of the functions given below graph the function and state the points at which it is continuous. (Use interval notation if needed. That is, if the function f is continuous at all a such that $-4 \leq a < 5$, you could write "f is continuous on $[-4, 5)$.") Also, if the given function has a discontinuity at a point $x = a$, state what sort of discontinuity it has there.

   (a) $\frac{2x^2-13x-7}{x^2-6x+9}$

   (b) $\frac{2x^2-5x-3}{x^2-6x+9}$

   (c) $\sin(x)$

   (d) $\sin(\frac{1}{x-1})$

   (e) $f(x) = \begin{cases} 3x+1 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0 \end{cases}$

   (f) $g(x) = \begin{cases} 3x+1 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0 \end{cases}$

2. Let's suppose that the function $f(x)$ is defined as a piecewise function:

$$f(x) = \begin{cases} -2x + c & \text{if } x \leq 3, \\ x^2 + 1 & \text{if } x > 3. \end{cases}$$

For some value of c the two "pieces" of the function will match up and form a continuous function. Find that value of c.

3. Repeat the previous exercise with the function

$$f(x) = \begin{cases} \sin(x) - 2 & \text{if } x \leq \pi, \\ 4x + c & \text{if } x > \pi. \end{cases}$$

4. Give the formula for a function that has infinite discontinuities at every one of the numbers $x = -7$, $x = -3$, $x = 0$, $x = 3$, and $x = 12$, but is continuous everywhere else.