Section 2.6: Limits, done right!

By now we've worked with ____________ quite extensively, so it's about time that we make sure we understand exactly what the *are*!

We'll give two definitions, the first of which will generally be enough to allow us to understand what's going on, and the second of which is far more technical and, as mathematicians would say, "rigorous."

"Soft" definition. We say that the ____________ *of* $f(x)$ *as* x *approaches* a is L, and we write

$$\lim_{x \to a} f(x) = \_\_\_\_ ,$$

if we can make $f(x)$ become as close to L as we like by choosing x as close as we like, but not equal to, ____ .

In a sense, then, if we write $\lim_{x \to a} f(x) = L$, we mean that $f(x)$ becomes "arbitrarily" close to L: no matter how close we want to make f to L, we can get there if we're willing to let x get close to a without actually reaching it.

In this sense, finding asserting the existence of a limit is like a *game*, with the following rules; if we claim that L is the limit of $f(x)$ as x approaches a, then...

1. if someone tells us *how close* they want us to make $f(x)$ get to L, we must
2. show them that if we put x close enough to a, $f(x)$ will *indeed* be as close as they wanted it to be to L.

Let's play this game with a really simple limit!

Example. Suppose that $f(x) = 2x + 5$ and $a = 3$.
In this case, what's $\lim_{x \to a} f(x)$ going to be, and why?

Good...now let's say that someone comes up to you and says "if that's really your limit, you need to show me that you can make $f(x)$ no further than $\frac{1}{2}$ from L."

Without skipping a beat, you respond: "if we make sure that x is no further than ____ from $a = 3$, then we're gold." What number can you put in the blank there, and why? (It might not hurt to draw a pretty decent graph in the space below; the more accurate, the better!)

Notice that there's a relationship between the slope of the function (which in this case was a line) and the distance we chose. Can you use this observation to respond to our challenger if she comes back and demands that $f(x)$ be no further than $\frac{1}{10}$ from 11?

How about no further than $\frac{1}{100}$? $\frac{1}{1000}$?

What would happen if we changed our function to $g(x) = 4x + 2$ and our point to $a = -1$? Suppose our challenger came at us with the numbers $\frac{1}{2}$, $\frac{1}{10}$, $\frac{1}{100}$, and $\frac{1}{1000}$ once more, how would you respond?

We can make the definition of the limit more precise by making clear exactly what's going on in this "challenging" process:

"Careful" definition. We say that $\lim_{x \to a} f(x) = L$ if for any real number $\epsilon > 0$ given to us, we can find another real number $\delta > 0$ such that if $0 < |x - a| < \delta$ (that is, as long as x at least "δ-close" to a without equalling a), then $|f(x) - L| < \epsilon$ (that is, $f(x)$ will be no further than ϵ from L).

If you unravel this definition carefully, you'll see it really is just a restatement of our game: our challenger comes at us with an ϵ, and we have to fend her off with a δ.

In the space below you should draw a graph illustrating this definition, ϵ and δ included!

Example. Use the *careful* definition of the limit to show that $\lim_{x \to 2} 3x - 1 = 5$. (Start off with "Suppose that $\epsilon > 0$ is given to us...")

Example. What about $\lim_{x\to 3} x^2$? Try to show that this limit is _____ !

In all of the examples that have come above, we were able to guess what the limit in question was because the limit we sought was the same as the value of the function at a:

$$\lim_{x\to a} f(x) = \_\_\_\_\_ \quad .$$

When this happens, we say that the function is _______________ . We'll have more to say about these functions in the next section, but for now we should consider the following

Example. In the space below, draw the graphs of a few different functions having the following characteristics: (1) f is continuous everywhere, (2) g is continuous everywhere except at $x = 2$, but $\lim_{x\to 2} g(x)$ can still be defined, and (3) h doesn't even have a limit at $x = 0$.

We'll be able to nail a good number of limits to the wall as soon as we understand continuous functions, but right now we can get a start on computing limits using a handful of pretty intuitively obvious rules, each of which has a nice paraphrasing that you can provide below:

Rules for limits.

1. $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a}$ _____

   Paraphrasing: "the limit of a sum is the _______ of the limits."

2. $\lim_{x\to a}(f-g)(x) =$

   Paraphrasing:

3. $\lim_{x\to a}(f\cdot g)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

   Paraphrasing: "the limit of a ____________ is the ____________ of the limits."

4. $\lim_{x\to a}\left(\frac{f}{g}\right)(x) =$

   Paraphrasing:

5. If c is a constant, then $\lim_{x\to a} cf(x) =$

   Paraphrasing: "the limit of a constant times a function is that constant...

There's one other important rule for limits of which we've already made use:

The Squeeze Theorem. Suppose that f, g, and h are functions such that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ and that for every x close to a we know $f(x) \le g(x) \le h(x)$. Then $\lim_{x\to a} g(x) =$ ____ as well.

Here's some space to draw a picture of the Squeeze Theorem:

When did this theorem come in handy already?

Homework. The following homework is due at 5:00 p.m. on *Friday, February 27th.*

1. For each limit given below, either guess what the limit is or explain briefly why the limit doesn't exist.

   (a) $\lim_{x\to 6} 4x - 5$

   (b) $\lim_{x\to 2} 7x^3 - 2x + 1$

   (c) $\lim_{t\to 0} \sin(t)$

   (d) $\lim_{x\to 0} |x|$

   (e) $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

   (f) $\lim_{r\to 0} e^r$

   (g) $\lim_{x\to \frac{\pi}{4}} \tan(x)$

2. Show carefully, using the rigorous definition of the limit, that $\lim_{x\to 4} 2x - 3 = 5$.

3. **(Extra credit.)** Show carefully, using the rigorous definition of the limit, that $\lim_{x\to 2} x^2 = 4$.

4. **True or false** (and please explain your answer!): if a is not in the domain of the function f, then $\lim_{x\to a} f(x)$ can *not* exist.