Section 2.5: The Product and Quotient Rules

We've started to stock our derivative tool chest, and by now we've got a good number of rules for derivatives. For instance, we know

1. **A power rule:**
2. **A rule for constant multiples:**
3. **A sum (or difference) rule:**
4. **A pair of trig derivatives:**
5. **A rule for exponential functions:**

The next couple of rules we'll uncover are much like the rule for sums above: they'll blow the lid off of our tool chest and give us an *enormous* number of new formulas.

The Product Rule.

Let's suppose we want to compute the derivative of the function $f \cdot g$ (which is defined "______________" by $(f \cdot g)(x) = f(x) \cdot g(x)$).

For ease of notation, let's just say that $u = f(x)$ and $v = g(x)$. We want to find $\frac{d(uv)}{dx}$.

Suppose we think of u as giving the *width* of a rectangle, and v as giving the *height* of the same rectangle. Then the product uv gives the _______ of the rectangle:

Thus asking for $\frac{d(uv)}{dx}$ is really asking how much the ________ of the rectangle changes if we change x just a little bit, and therefore change both u and v.

That is, $\frac{d(uv)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta(uv)}{\Delta x}$. In the space below, you can draw a picture of what happens if we change x by a little bit, Δx, giving us a slight change, Δu and Δv respectively, in both u and v:

What extra area do we get, on top of the original area? That is, since

$$\Delta(uv) = (\text{new area}) - (\text{old area}),$$

what's $\Delta(uv)$?

Now divide this by Δx, and imagine letting Δx get as small as you'd like (*i.e.*, take the limit as $\Delta x \to 0$). What results?

When you translate it into statements about the derivatives of f and g, this gives the following

Theorem (Product Rule). $\frac{d}{dx}(f \cdot g) =$ ____________________ .

Examples. Find the derivatives of each of the following functions.

1. $f(x) = x^2 \sin(x)$

2. $g(x) = xe^x$

3. $h(r) = r\cos(r)$

4. $K(x) = xe^x \sin(x)$ (*Hint:* use the Product Rule twice!)

An intermediate derivative.

No matter what else it does clumsily, I like how your textbook handles this next step.

Proposition. Let f be a function. The $\frac{d}{dx}\left(\frac{1}{f}\right) = \frac{-df/dx}{f^2}$.

Proof. We know that $f \cdot \frac{1}{f} = 1$. Now let's just differentiate both sides and solve for $\frac{d}{dx}\left(\frac{1}{f}\right)$:

The Quotient Rule.

The whole reason for that last intermediate step was to prove the

Theorem (Quotient Rule). $\frac{d}{dx}\left(\frac{f}{g}\right) = \underline{\hspace{6cm}}$.

Proof. Writing $\frac{f}{g}$ as $f \cdot \frac{1}{g}$, we can combine the Product Rule with the intermediate proposition above. Here's some room for the cool computations:

Yowza! This means that now we can compute derivatives not only of products, but also of quotients!

Examples. Find the derivative of each of the following functions.

1. $f(x) = \frac{x}{\sin(x)}$

2. $g(x) = \tan(x)$ (*Hint:* $\tan(x) = \frac{\sin(x)}{\cos(x)}$.)

3. $H(\theta) = \frac{e^\theta}{\theta^2}$

How could we have computed the preceding derivative without using the Quotient Rule? Use this alternative method to find the derivative *again*, in the space below:

Homework. The following homework is due at 5:00 p.m. on *Friday, February 20th.*

1. Find the derivative of each of the following functions.

   (a) $\sec(x)$

   (b) $\frac{x \sin(x)}{e^x}$

   (c) $\theta^n e^\theta$ (here n is a constant)

   (d) $\sin^2(x)$ (Recall that $\sin^n(x)$ is shorthand for $(\sin(x))^n$.)

   (e) $\sin^3(x)$ (*Hint:* use the previous computation...)

   (f) $\sin^4(x)$

   (g) $\frac{3x^2-x+1}{x^5+7x+1}$

   (h) $\frac{\tan(x)+\sin(x)}{e^x+\cos(x)}$

2. Come up with two functions f and g that demonstrate that the "Fake Product Rule" $\frac{d}{dx}(f \cdot g) = \frac{df}{dx} \cdot \frac{dg}{dx}$ is ***false***. (Please show your computations!)

3. Come up with two functions f and g that demonstrate that the "Fake Quotient Rule" $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'}{g'}$ is ***false***. (Please show your computations!)