Section 2.4: Differentiating trig functions

The goal of today's class is to nail down one particularly difficult derivative:

$$\frac{d}{dx}(\sin(x)) = \cos(x).$$

Let's dive right in, shall we?

The first computations we perform are quite straightforward: we simply apply the definition of the derivative to $f(x) = \sin(x)$ and see how far we can get using trig identities:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \cdots \end{aligned}$$

Notice that we used the law $\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$ in the above computations. Everything therefore comes down to computing the following two limits:

$$\lim_{h\to 0} \frac{\sin(h)}{h} \quad \text{and} \quad \lim_{h\to 0} \frac{\cos(h) - 1}{h}.$$

Let's use *Mathematica* to get a reasonable guess as to what this first limit might be. Put your guess in the space below:

To *prove* this limit we argue geometrically. The following diagram will help us out *immensely*:

Step 1. By the definition of radian measure, the length of the arc AB in the above diagram is θ. Furthermore, since $|OB| = 1$ (why is this?), you should be able to show that $|BC| = \sin\theta$:

Well, how does the length $|BC|$ compare with the length of the arc AB, and what does this say about $\sin\theta$, and therefore about $\frac{\sin\theta}{\theta}$?

Step 2. Now look that the segments AE and BE of the polygon circumscribing the circle of which AB is an arc. Which is greater, the length of AB or the sum of the lengths of these segments?

Also, we can get an estimate on the sum of the lengths of these segments:

$$|AE| + |BE| < |AE| + |DE| = |AD|.$$

But what's this last length, in terms of the angle θ?

All told, we get what inequality involving θ?

Step 3. Let's put the *squeeze* on! We've now got the following arrangement:

$$\cos\theta < \_\_\_\_ < \_\_\_\_ \quad .$$

As θ gets closer and closer to 0, what can you say about the first and the last terms in this set of inequalities?

Thus

$$\lim_{h\to 0} \frac{\sin(h)}{h} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = \_ \quad .$$

With a little trickery we can establish the value of the other limit we're interested in. Try multiplying both the top and the bottom of $\frac{\cos(h)-1}{h}$ by the "conjugate" $\cos(h)+1$ and following your nose:

Whew! Plugging all of this into our earlier computations for $\frac{d}{dx}\big(\sin(x)\big)$, we now have a formula:

$$\frac{d}{dx}\big(\sin(x)\big) = \_\_\_\_\_\_ \quad .$$

Similar computations yield the following formula:

$$\frac{d}{dx}\big(\cos(x)\big) = \underline{\hspace{3em}} \quad .$$

In a little while we'll learn a rule that will let us compute the derivatives of all of the other trig functions!

Finally, I'll note that knowing the limit $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ allows us to compute related limits:

Example. Compute the limit $\displaystyle\lim_{x\to 0} \frac{\sin(7x)}{4x}$.

This follows pretty easily once we do all that we can to make the $4x$ in the denominator look like $7x$:

Homework. As *I* will now be writing many of our homework problems, I will type those up on a separate sheet of paper, available on the class website. The homework for this section will be due at 5:00 p.m. on *Friday, February 13th.*