Section 2.2: Derivatives of polynomials (and some other stuff along the way)

Right now we should feel pretty confident in stating the following theorem, which we've proven in the cases $n = 1, 2, 3, -1$, and $\frac{1}{2}$:

Theorem. Let $f(x) = x^n$ for some real number n. Then $f'(x) = nx^{n-1}$.

Let's see if we can prove this is true for *all* positive integers $n = 1, 2, 3, ...$, in one fell swoop.

Step 1. Recall that on Friday we talked about the following fact:

Proposition. Let n be a positive integer. Let $\binom{n}{k}$ denote the fraction $\frac{n!}{(n-k)!k!}$, where $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$ and $0! = 1$. Then

$$(x+y)^n = \sum_{k=1}^{n} \binom{n}{k} x^{n-k} y^k = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

You might recall that we can recover these numbers $\binom{n}{k}$ from Pascal's triangle, the first few rows of which you can fill in in the space below:

We'll really only need to focus on the *second* entry in each row: what is $\binom{n}{1}$, no matter what n is?

Step 2. Let $f(x) = x^n$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

Using Pascal's triangle, fill in the blank coefficients in the expanded version of $(x+h)^n$ given below:

$$(x+h)^n = \_\_\ x^n + \_\_\ x^{n-1}h + \cdots + \_\_\ xh^{n-1} + \_\_\ h^n.$$

Step 3. The last thing you wrote down was one term in the numerator of the fraction we're trying to compute. In the space below, subtract off the other term and divide by h to find the fraction itself:

Step 4. We can now let $h = 0$! The result is the derivative:

Pretty neat, huh?

There's one special case I'd like to call your attention to:

Fact. The derivative of a constant function is 0. That is, if $f(x) = c$, then $f'(x) = 0$.

Why does this make sense *graphically*?

Here's another nice fact about derivatives:

Proposition. If c is a constant, then the derivative of the function $cf(x)$ is $cf'(x)$. Written in Leibniz notation, this is $\frac{d(cf)}{dx} = c\frac{df}{dx}$.

Proof. The small space below should suffice:

One more! Then we'll be ready to put it all together and say something even cooler:

Theorem. The derivative of the sum of two functions is the sum of their derivatives. That is, $(f+g)'(x) = f'(x) + g'(x)$. In Leibniz notation, this is $\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$.

Proof. Just as we did in the last few proofs, start off with the definition of the derivative of $f+g$. I'll start you off:

$$(f+g)'(x) = \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = \cdots$$

Here we go! A *general* mathematical statement involving derivatives:

Theorem. Let $f(x)$ be a polynomial, so that $f(x)$ has the form

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n.$$

Then

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n-1) a_{n-1} x^{n-2} + n a_n x^{n-1}.$$

Note. This theorem subsumes all of the previous computations we've done, including powers, constant multiples, *and* sums!

Proof. This is really just a matter of putting together the previous propositions and theorems. Let's use the Leibniz notation to see how neatly it breaks up:

$$\frac{df}{dx} = \frac{d}{dx}(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n) = \cdots$$

That's all we'll need for now. During the next class we'll consider some applications: where are derivatives *used*?

Homework (*Due Friday, February 6th*) Do exercises 3-11, 14, 16, 19, 31, and 35 from Section 2.2 (pages 56-57) of your text.