MATH 191, Section 2
Calculus I
Spring 2009

Practice Exam 3 Solutions

This practice exam is similar in length, content, and format to the actual exam. This is not to say that the problems given here represent *all* of the concepts you will encounter on the actual exam, since it's difficult to "cover" all possible subjects in such a short exam! However, if you feel confident on your performance on this practice exam and you've gone over all homeworks and quizzes, you should feel confident about your upcoming performance on the actual exam.

In order to save paper, I have not included space for you to work out your solutions. (The actual exam will provide such space.) Rather, please complete solutions to the below problems on your own paper. The practice exam is worth a total of 100 points; the point value of each question is provided with that question.

1. (18 points total; 6 points each) Evaluate each of the following limits.

   (a) $\lim_{x\to\infty} \frac{3x^4+3x-1}{7x^4-3x^3+x^2-x}$

   We use L'Hôpital's Rule at each step, obtaining new limits until we have an "obvious" final limit:

$$\begin{aligned}\lim_{x\to\infty} \tfrac{3x^4+3x-1}{7x^4-3x^3+x^2-x} &= \lim_{x\to\infty} \tfrac{12x^3+3}{28x^3-9x^2+2x-1}\\ &= \lim_{x\to\infty} \tfrac{36x^2}{84x^2-18x+2}\\ &= \lim_{x\to\infty} \tfrac{72x}{168x-18}\\ &= \lim_{x\to\infty} \tfrac{72}{168}\\ &= \tfrac{72}{168} = \tfrac{3}{7}.\end{aligned}$$

   (b) $\lim_{x\to\pi} \frac{x-\pi}{\cos(x)}$

   Be careful! LHôpital's Rule doesn't apply here, since although the numerator approaches 0, the denominator approaches $\cos(\pi) = -1$, so we don't need L'Hôpital's Rule: the limit is just $\frac{0}{-1} = 0$.

   (c) $\lim_{t\to\infty} \frac{\ln(t)}{\sqrt{t}}$

   Here we use L'Hôpital's Rule again, since we have an indeterminate form of type "$\frac{\infty}{\infty}$":

$$\lim_{t\to\infty} \frac{\ln(t)}{\sqrt{t}} = \lim_{t\to\infty} \frac{1/t}{\frac{1}{2\sqrt{t}}}.$$

   Before you go any further, simplify what you've just obtained:

$$\lim_{t\to\infty} \frac{1/t}{\frac{1}{2\sqrt{t}}} = \lim_{t\to\infty} \frac{1}{t}\cdot\frac{2\sqrt{t}}{1} = \lim_{t\to\infty} \frac{2\sqrt{t}}{t} = \lim_{t\to\infty} \frac{2}{\sqrt{t}}.$$

   Since $\sqrt{t} \to \infty$ as $t \to \infty$, the denominator grows without bound and the numerator is a finite constant. Thus the limit is 0.

2. (14 points) Two berserker guinea pigs begin running from the same point at the same time. One runs due west at 1 km/hr, and the other runs due south at 2 km/hr. How fast is the distance between the two guinea pigs changing 1 hour after the time at which they began running?

The key to this problem is to note that the pigs are traveling at right angles to one another, so the distance between them is the length of the hypotenuse of a right triangle. The legs of this triangle have lengths equal to the distances the pigs have traveled at a given point in time.

Let's denote by x the distance the westward-moving pig has gone, and by y the distance the southward-moving pig has gone. (Both are functions of time t.) Thus the distance $D(t)$ between the pigs at time t is given by $D^2 = x^2 + y^2$. We are given $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 2$, and we need to find $\frac{dD}{dt}$ at time $t = 1$.

To get a relationship between the rates of change we know and the rate of change we're trying to find, differentiate with respect to t:

$$D^2 = x^2 + y^2 \Rightarrow 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2x \cdot 1 + 2y \cdot 2 = 2x + 4y.$$

At time $t = 1$, $x = 1$ and $y = 2$, so $D = \sqrt{1^2 + 2^2} = \sqrt{5}$, and the last equation gives us

$$2\sqrt{5}\frac{dD}{dt} = 2(1) + 4(2) = 10.$$

Thus $\frac{dD}{dt} = \frac{10}{2\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}$ km/hr.

3. (15 points total) Let $f(x) = 3x + x^2 - x^3$.

   (a) (5 points) Explain carefully why the Mean Value Theorem applies to the function f on the interval $[-1, 2]$.

   Because the function is a polynomial, we know that it's always differentiable and continuous, and so in particular is differentiable and continuous on the interval $[-1, 2]$. Thus the MVT applies with no hitches!

   (b) (10 points) Find all values c making the Mean Value Theorem true for f on the interval $[-1, 2]$.

   We need to find all c on the given interval such that $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(-1)}{2-(-1)} = \frac{3}{3} = 1$. Since $f'(x) = 3 + 2x - 3x^2$, we must solve the equation $-3c^2 + 2c + 3 = 1$, or $-3c^2 + 2c + 2 = 0$.

   Use the quadratic formula:

$$-3c^2 + 2c + 2 = 0 \Rightarrow c = \frac{-2 \pm \sqrt{4 - 4(-3)(2)}}{-6} = \frac{-2 \pm \sqrt{28}}{-6} = \frac{2 \pm 2\sqrt{7}}{6} = \frac{1 \pm \sqrt{7}}{3}.$$

   Since both of these values lie on the interval $[-1, 2]$, we'll take them both!

4. (21 points total; 7 points each) Find the derivative of each function given below.

   (a) $f(t) = \sin^{-1}(e^t)$

   Applying the Chain Rule along with the formula for the derivative of $\sin^{-1}(x)$, we obtain

$$f'(t) = \frac{1}{\sqrt{1 - (e^t)^2}} \cdot \frac{d}{dt}\left(e^t\right) = \frac{e^t}{\sqrt{1 - e^{2t}}}.$$

(b) $g(x) = e^{\sin(x)}$

This is just Chain Rule:

$$g'(x) = e^{\sin(x)} \cdot \frac{d}{dx}(\sin(x)) = e^{\sin(x)}\cos(x).$$

(c) $h(x) = \ln(x\sin(x))$

Now we combine the Chain Rule with the Product Rule:

$$h'(x) = \frac{1}{x\sin(x)} \cdot \frac{d}{dx}(x\sin(x)) = \frac{1}{x\sin(x)}(1 \cdot \sin(x) + x \cdot \cos(x)) = \frac{\sin(x) + x\cos(x)}{x\sin(x)}.$$

5. (10 points) Find the equation of the tangent line to the graph of the relation $x^2y^3 - x - 1 = y$ at the point $(2, 1)$.

The first goal is to find the slope of the tangent line we seek, for which we need the derivative $\frac{dy}{dx}$. We can find this derivative implicitly, differentiating both sides of the equation above and thinking of y as a function of x (don't forget the Chain Rule!):

$$\begin{aligned} \tfrac{d}{dx}\left(x^2y^3 - x - 1\right) = \tfrac{dy}{dx} \quad &\Rightarrow \quad 2x \cdot y^3 + x^2 \cdot 3y^2\tfrac{dy}{dx} - 1 = \tfrac{dy}{dx} \\ &\Rightarrow \quad 2xy^3 - 1 = \tfrac{dy}{dx} - 3x^2y^2\tfrac{dy}{dx} \\ &\Rightarrow \quad 2xy^3 - 1 = \tfrac{dy}{dx}\left(1 - 3x^2y^2\right) \\ &\Rightarrow \quad \tfrac{dy}{dx} = \tfrac{2xy^3-1}{1-3x^2y^2}. \end{aligned}$$

Now all we have to do is plug in the values we're given for our point on the graph. Letting $x = 2$ and $y = 1$, we get the slope at the point of interest:

$$m = \frac{2xy^3 - 1}{1 - 3x^2y^2} = -\frac{3}{11}.$$

Finally, we can use the point-slope formula for a line to determine the equation of the tangent line:

$$y - y_1 = m(x - x_1) \Rightarrow y - 1 = -\frac{3}{11}(x - 2) \Rightarrow y = -\frac{3}{11}x + \frac{17}{11}.$$

6. (12 points total; 6 points each) Let $f(x) = x^3$.

(a) Draw a graph which illustrates carefully how to use 4 rectangles to approximate the area under the graph of f between $x = 0$ and $x = 1$.

Your graph should show 4 rectangles giving an overestimate for the area we're trying to compute, sitting over intervals with endpoints $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1.

(b) Find the approximation to the area under f given by the 4 rectangles you drew in (a).

Each of the rectangles you drew has width $\frac{1}{4}$, and the heights of the rectangles (from left to right) are given by plugging $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1 into $f(x) = x^3$. Thus the rectangles' heights are $\frac{1}{64}$, $\frac{1}{8}$, $\frac{27}{64}$, and 1, from left to right. The approximation to the total area is now easy to find:

$$A = \frac{1}{4} \cdot \frac{1}{64} + \frac{1}{4} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{27}{64} + \frac{1}{4} \cdot 1 = \frac{1 + 8 + 27 + 64}{256} = \frac{100}{256} = \frac{25}{64} = 0.390625.$$

7. (10 points) Show how to find the derivative of $y = \sin^{-1}(x)$ using implicit differentiation.

Since $y = \sin^{-1}(x)$, $x = \sin(y)$, and we differentiate this last equation implicitly:

$$1 = \cos(y) \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}.$$

Now it's useful to recall that $\sin^2(A) + \cos^2(A) = 1$ gives us $\cos(A) = \sqrt{1 - \sin^2(A)}$, so applying this formula with $A = y$ we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2(y)}} = \frac{1}{\sqrt{1 - x^2}}.$$